Master's degree thesis

LOG950 Logistics

Determining optimum order size for family of items under planned stock out situation and capital constraint using cost minimization approach

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Preface

Above all, I would like to express my sincere gratitude to my advisor Associate Prof. Øyvind Halskau for his continuous support of my master's thesis research, for his patience, motivation, and critical comments. His guidance helped me in all the time of research and writing of this thesis.

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My sincere thanks also go to my parents and other friends who assisted and motivated me in this research.

Summary

Traditional cost minimization approach to inventory control exclude the possibility of back orders. It also excludes the possibility of capital (budget) constraint. This master's thesis included the concept of planned stock out (back ordering) and capital constraint and it found a formula to find the optimum order size and optimum percentage of back order that minimizes total relevant inventory costs for family of items. It assumed all stock out situations will be back ordered so that there will not be lost sales. The paper is organized in to six parts. The first part is introduction that discussed three approaches to inventory control and the general objective of the research. The second part presented an overview of basic concepts and related literatures. In the third part the mathematical models for family of items are formulated and formulas for a single item case are discussed. The forth part solved the mathematical models formulated for family of items in the third part are solved and analyzed. The fifth part presented numerical example based on the formulas found in part four. Finally summary and recommendations for further research are presented.

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CHAPTER 1

1. INTRODUCTION AND RESEARCH OBJECTIVE

1.1. Introduction

Inventory is one of the major and costly investments of many companies and there exist different types of costs that are incurred when a company decides to have an inventory. The cost elements include purchasing costs, order costs, cyclic inventory holding costs, cost of safety stocks, stock out costs and cost of backordering. Inventory is also a major use of working capital. Inventory to asset ratio has been used to show the capital tied up in inventory. The ratio varies from company to company. For some companies this ratio reaches up to 40 % (Silver, Pyke and Peterson 1998). Because the inventory requires significant amount of capital, it is important to control the capital tied up in inventory (Axsäter 2006).

The reason for this is of course the capital tied up in the inventory has an alternative use. Companies want to use their scares capital in an optimal way. Keeping inventory at low level helps to make cash available to other company's activities, minimize the cost of holding high level of inventory. On the other hand keeping high level of inventory enable a company to have high level of customer service, reduce the possibility of production stops due to shortage of raw material inventory, decreases ordering cost and have an advantage of getting quantity discount on large volume purchase.(Axsäter 2006). There is a trade off between advantage of holding large amount of inventory and the cost associated with it. This shows that one of the main inventory management decisions is to find the optimum level of inventory which balances the tradeoff between holding cost, carrying cost and stock out cost so that to minimize the total inventory cost. On the other hand inventory is an asset and it is considered as an investment. Stock holders are interested on the return of their investment on inventory. In this case inventory shall be managed with the objective of maximizing the return on investment.

According to Silver, Pyke and Peterson (1998) inventory management decision involves mainly with deciding on:

How often inventory status should be determined

When order replenishment should be done and

How large the replenishment order should be.

The objective of these decisions is efficient and effective utilization of inventory and the capital tied up in it. This can be measured using the objective of cost minimization, profit maximization, return on investment maximization or other approached that can be used as performance measure.

Companies can determine the level of inventory either on continuous bases (Perpetual inventory system) or based of a fixed time interval (periodic inventory system). The continuous bases determine the availability and the quantity of inventory continuously as function of doing business. When the inventory level reaches at a given minimum amount (reorder point) a new order is placed. The size of the order is constant in this case. The periodic inventory system determines the level of inventory every fixed time interval and a new order size that increase the inventory to the given maximum level is placed. Here the size of order placed each time varies depending on the amount of inventory left in stock. In this masters thesis the perpetual inventory system is assumed where a fixed amount of inventory is ordered each time the level of inventory reaches at reorder point.

Determining the exact size of the replenishment quantity is associated with finding the order size that can optimize the cost, profit or return on investment objectives. The time when inventory is replenished is related with the time it takes to receive order. The time between making an order and receiving the order is called order lead time.

There are different inventory models that have been used to determine the optimum size of replenishment order. The most common approach used to determine order size and replenishment frequency has been the classical economic order quantity model. The classical approach shows how we can balance the various costs of stock to answer the question ‘how much should we order?’

The approach is to build a model of an idealized inventory system and calculate the fixed order quantity that minimizes total cost (Waters 2003). This optimal order size is called economic order quantity (*EOQ*). It is based on the objective of minimizing the total relevant cost such as ordering and holding costs. This model was introduced by Harris in 1915, but the calculation is often credited to Wilson (1934) who independently duplicated the work and marketed the results. This Model is based on the following assumptions:

The demand rate is constant and deterministic

The order quantity need not be an integral number of units and there are no minimum and maximum restrictions on its size

The unit variable cost does not depend on the replenishment quantity (that is, there are no discounts in either the unit purchase cost or the unit transportation cost)

The cost factors do not change appreciably with time (that is, inflation rate is low)

The item is treated entirely independently of other items (that is, benefits from joint review or replenishment do not exist or are simply ignored)

No shortages are allowed

The entire order quantity is delivered at the same time

The product is always available in the market

The most important assumption is that demand is known exactly and constant over time. This and other assumptions seem unrealistic, but *EOQ* model has been widely used as a good approximation to reality. In a continuous demand the stock level decline steadily over time. Constant demand means that the decline is always at the same rate. If the lead time is zero; we do not place an order before stock actually runs out. The assumption that no shortage is allowed means that the stock level never falls below zero and there are no lost sales.

The objective of classical optimization model is minimizing the total relevant costs. Relevant costs are costs that are truly affected by the choice of order quantity. The unit purchasing cost is assumed to be constant. This means that purchasing cost per unit cannot be affected by the size of order quantity and the total purchasing cost become the same regardless of order size. Therefore, it is not part of total relevant cost. Total ordering cost and carrying cost are affected by the order size. Order size affects the total number of order per year and the level of inventory, in units. If the order size is large, the total numbers of orders per year become fewer and results in lower ordering cost. The impact of large order size on holding cost is the opposite. Large order size increases the average inventory level and results higher carrying cost. Therefore ordering cost and holding cost are considered as the only relevant costs by the classical optimization approach.

In the basic Harris-Wilson case, the annual total inventory cost is the sum of purchasing costs, ordering costs and holding costs. However, it is only the ordering cost and holding cost that are affected by the choice of order quantity. This is because the purchasing costs do not vary based on the quantity ordered and also the annual demand is known and constant. So the total relevant cost is a function of ordering costs and holding costs. Total relevant costs are depicted by the following formula:

$$TRC = \frac{DA}{Q} + \frac{Qvr}{2}. \qquad (1.1)$$

To obtain the minimum cost order size (*EOQ*), take the first derivative of total relevant cost function with respect to the order size (*Q*) and set it equal to zero:

$\frac{dTRC(Q)}{\delta Q} = \frac{vr}{2} - \frac{AD}{Q^2} = 0$ and the second derivative ($\frac{d2\ TRC(Q)}{d2Q} = \frac{2AD}{Q^3}$) is always positive and hence we have a minimum point. The order size that minimizes the total relevant cost is given by the formula below.

$$EOQ^* = \sqrt{\frac{2DA}{vr}} \qquad (1.2)$$

Where

Q = order quantity or lot size in units

D = annual demand in units

A = ordering cost per order

v = unit variable cost of the item

r = th*e* carrying charge, the cost of having one dollar of the item tied up in inventory for a unit time interval (usually one year.)

What we have looked at so far is to consider stable and unchanging demand conditions where the objective is to minimize the total relevant logistics cost over the long term. However, there are situations that need short term decisions, in extreme cases for a single period when the demand is discrete and uncertain. In discrete and stochastic demand making both large and small orders has its own trade off. The trade off is between the risk of overstocking (disposing with less than the purchasing cost) and risk of under stocking (losing the opportunity of making profit). Newsboy problem is one of the models used to aid decision making when demand is discrete and stochastic. The news boy problem is used to decide how many units to buy when there is a single purchasing opportunity before the start of the selling period. For example if the newsboy buys too few papers he has unsatisfied demand which could have given a high profit and if he buys too many papers he is left with unsold stock which has no value or value less than the purchasing cost at the end of the day. The objective of newsboy problem is to find the order quantity that maximizes the expected profit.

The profit for a given Q and x units is given below:

$$\pi(Q,x) = pQ - vQ - B(x-Q) \;\; if\, Q \leq x$$

$$\pi(Q,x) = px - vQ + g(Q-x) \;\; if\, Q > x$$

Where,

x is stochastic demand

Q is order size

P is the unit selling price

v is the unit purchase price

g is the unit salvage value

B is the unit stock out cost

However, stochastic demand and newsboy problem are not the interest of the researcher and are not dealt in detail in this thesis.

Most of the literature in the past discussed cost minimization or profit maximization as an optimizing criterion. But there have been few researches that tried to use return on investment (*ROI*) as an approach to determine optimal order quantity. This approach tries to maximize the return on investment. Return on investment is the ratio of gross profit to owners' equity (Trietsch 1995). It is a quotient profit measure expressed as the ration of gross profit to the capital utilized to obtain the profit (Halskau and Thorstenson 1998). It relates income with resources as a measure of performance. When viewed from the standpoint of the owner or investor, maximizing the return on investment (*ROI*) is an appropriate criterion for many types of inventories. This approach views inventory as an investment. In this case one can express return on investment as the ration of gross income to average inventory. From the point of view of the owners return on investment is a better approach to evaluate performances (Schroeder 1995).

ROI = Income/investment = $\frac{\text{gross profit} - \text{total logistic cost}}{\text{average inventory}}$

The assumption is that the only cost is the ordering and holding costs as specified in the classical cost minimization approach. Mathematically Schroeder (1995) expressed return on investment as shown below.

$$ROI = \frac{D(p-v)-(QA/D+Qvr/2)}{(Qv)/2}$$

Where;

D = annual demand

p = selling price

Q = order size

v = purchasing price

r = carrying charge

To obtain the order size, take the first derivative of ROI function with respect to the order size (Q) and set it equal to zero. Schroeder (1995) found the formula given below to compute the optimum order size.

$$Q^* = 2A/(p-v)$$

The above formula shows that order size is only affected by ordering cost and the gross profit margin. The interest rate (r) and the size of annual demand do not have any impact on order size. Ordering cost and optimum order quantity have direct relationship. When ordering cost increase, the order quantity also increase and vise versa. The mark-up $(p-v)$ has indirect relationship with the optimum order quantity. The higher the mark-up, the lower is the order size. When the mark-up decreases the order size increases.

The return on investment function above does not allow shortages. This is because the assumptions are adopted from the classical cost minimization model. The classical optimization models usually assume that no shortages are allowed and that all demand must be met from the shelf. In this case the shortage cost is so large that any shortage would be prohibitively expensive. Usually this happens when the cost of keeping an item in stock is higher than the profit from selling it (Waters 2003). There are, however, circumstances where planned shortage is beneficial.   In markets like information technology (IT), consumer electronics and even in automobile, some customers show the willingness to wait until delivery (Che, Narasimhan and Padmanabhan 2010). For example when you buy a new car the show room does not keep a stock of every all models, but you choose the feature you want, then show room orders this from the manufacturer, and then you wait for the car to be delivered. When customer demand for an item cannot be met from stock, there is shortage. In this situation the customer has two choices. The first is they can wait for the item to come to stock, i.e. the demand is met by back-ordering. There are always costs associated with backorders such as administrative costs, loss of goodwill, some loss of future orders, emergency orders, expediting and so on (Waters 2003). The second choice is that the customer can withdraw his or her order and go to a competitor, in which case there are lost sales. In this case all stock outs (shortages) are lost and not recovered. Customers may transfer all future business to other suppliers. This has a direct negative impact on a company's market share and future existence.

1.2. Objective of the research.

There are situations where customers develop more faith and loyalty to their suppliers. In this case customers, when shortage occurs, are willing to wait for back-orders. However, customers have their own level of patience and if they loose their patience they may turn to other suppliers and the back-order can be changed to lost sales. The behavior of customers faced with shortage depends on the waiting time until the arrival of the next order and on the time that has elapsed since the stock out happened (Sicilia, San-Jose, and Garcia-Laguna 2009).

There exist many researches that tried to extend the classical optimization model by introducing the concept of lost sales and back-ordering. Most of the inventory models presented a situation, during stock out, where a fraction of demand is back ordered and the remaining fraction is lost sales. The assumption is some customers are patient to wait until their demand is met from back order, while the other impatient customers fulfilled their demand from other suppliers. This means that all shortages cannot be back ordered. However, in this research demand is assumed to be constant and known and all shortages will be back ordered. It excludes the possibility of lost sales. It is a situation where part of customer order is fulfilled directly from the shelf and all the shortages are satisfied from back order.

Usually there is a limit on the availability of fund to buy inventory that makes the objective a constrained cost minimization objective. This research included the concept of planned stock out and capital constraint to the classical cost minimization approaches. It considered continuous review inventory model with complete back-ordering (there is stock out and customers are willing to wait their order be fulfilled from a new order arrival) and budget constraint. The objective is to determine the economic order size and proportion of back-order for a family of items under planned stock out situation and capital constraint. In this research the word planned stock out and planned back order are used interchangeably. You will see the details of the assumptions and the objective in the model formulation in chapter three.

CHAPTER 2

2. BASIC CONCEPTS AND RELATED LITERATURE REVIEW

2.1. Basic concepts

2.1.1. Definition of Inventory.

Many organizations store materials until they are needed for production or sale. For example a wholesaler buy products from manufacturers and stock until it sells to retailers; a factory keeps a stock of raw materials for its products; a bank holds cash for its day-to-day transactions. When an organization has materials that it does not use immediately, it puts them in to stock.

Stock consists of all the materials and products stored and kept for future use while an inventory is a list of the items held in stock (Water 2003). Recently it is becoming more common to use the name inventory for both the list of inventory and the stock itself.

“Inventory may consist of supplies, raw materials, in-process goods and finished goods” (Tersine 1994, 3)

Supplies are materials that will not be part of the final products rather they are consumed in the process of transforming raw materials in finished goods. Raw materials are items that will be transformed in to finished good by the production processes. Work -in-process (WIP) inventory is an item that is partly completed but still in the production process for completion. Finished goods are items that are completely transformed in to end product and available for sale, distribution and store.

Silver, Pyke and Peterson (1998) classified inventories on functional bases. They recommended six different categories of inventories: cycle stock, congestion stock, safety stock, anticipation inventories, pipeline inventories and decoupling stocks.

Cycle inventories. Companies usually make order in batches rather than ordering single units. This is mainly because of the advantages associated with batch order such as economies of scale (due to large set up cost) and quantity discount on purchase price and transportation cost. The amount of inventories left on hand is called Cycle stock. The size of cycle stock depends on the size of batch order. The management decides the order size by taking in to consideration the trade off between holding costs and ordering cost.

Congestion stock. This usually happen in production process. When items share the same production equipment, inventories build up as they wait until the equipment is available. The inventory built up in this way is called congestion stock.

Safety stock. Safety stock is kept to buffer the variation in demand and supply. The level of safety stock is dependent on customer service level set by management of a company. It is the management who decides the customer service level. The higher the customers service level, the higher will be the level of safety stock. The lower the service level, the lower will be the level of safety stock. Safety stock is held to reduce the possible lost sales and customers' dissatisfaction

Anticipation inventories. Demand rate may not be the same through the year. It may be lower than the average in some part of the year. During this period stock is accumulated to meet anticipated peak demand. So it is stock accumulated in advance to meet the demand surge in peak sales time.

Pipeline inventories. Theses are inventories (work- in- process inventories) that are in transit between two adjacent locations or goods in transit in Trucks, Rail or pipeline.

Decoupling stock. Decoupling inventory is used to prevent breakdown or interruption of production process between two or more interdependent operations. It is a shared inventory by these interdependent divisions. It reduces output synchronization.

Acquiring and holding the different type of inventory has different associated costs. The next topic discusses the different types of inventory costs.

2.1.2. Types of inventory cost

Inventory cost includes purchasing cost, holding cost, ordering cost and stock out cost. Purchasing cost is the units purchase price (if purchased from external source) or unit production cost (if produced inside the organization). It is denoted by v and is it is usually expressed in monetary terms.

Holding cost includes such costs associated with investing in inventory and maintains the inventory in store. Cost of capital, insurance, taxes, deterioration, storage and obsolesce are examples of holding cost and opportunity cost of the money invested (Silver, Pyke and Peterson 1998). Carrying cost is usually computed per year.

Carrying cost per year = $\frac{Q}{2}vr$

Where $\frac{Q}{2}$ is the average inventory in units, *r* is carrying charge, the cost in dollar of carrying one dollar of inventory for one year and *v* is unit purchasing price. According to Silver, Pyke and Peterson (1998) the largest percentage of carrying charge is the opportunity cost of warehouse space and capital tied up by the inventory. Opportunity cost is the return on investment that could be earned by investing the money tied up in inventory to the next best investment opportunity. The cost of capital is dependent on the level of risk, and inventory investment is considered as low risk which in turn make the cost of capital a relatively small(Silver, Pyke and Peterson, 1998). Of course the value of *r* is not only dependent on the cost of capital, but also on the cost of storage.

Ordering cost is cost associated with making requisitions, analyzing vendors, writing purchase orders, receiving materials, inspecting materials, following up orders and making sure that the transaction is completed are part of ordering cost. It is assumed to be independent of the size of the replenishment. It is usually denoted by the letter *A*.

Stock out cost is cost associated with failing to meet customers' orders. It is the cost incurred when a stock out occurs. Lose of good will, lost sales, cost of backordering, emergency shipment or substitution of less profitable item are the main examples of stock out cost. When an item is temporarily out of stock two extreme cases may happen to customer's order: complete back-ordering or complete lost sales (Silver, Pyke and Peterson, 1998).

Complete back ordering occurs when there is stock out and customers are willing to wait their order be fulfilled from a new order arrival. This situation is common in government organization and wholesale- retail distribution system. In this case stock will be negative between the time the stock out situation occurred and the next replenishment arrives.

Complete lost sales. In this case customers go to other competitors to satisfy their need. Customers cannot wait until a new order has arrived to satisfy their demand. This is most common in retailer-consumer link

If holding inventory has its own cost, why organizations hold inventory? Economies of scale and uncertainty of demand and supply are the cause to hold inventory (Axsäter 2006). According to Tersine (1994) there are four functional factors that force organizations to hold inventory. These are time factor, discontinuity factor, uncertainty factor and the economy factory.

If companies wait customers' order to start production, it is not possible to deliver costumers' order immediately. This is because it takes time to produce and distribute goods to final consumers. This time factor is one of the reasons to hold inventory so that organizations can reduce the lead time in meeting customer order.

The discontinuity factor allows the treatment of various dependent operations (retailing, distributing, warehousing, manufacturing and purchasing) in an independent and economic manner. The uncertainty factors include error in demand estimates, variable production yields, and equipment breakdowns, shipping delay, strike and bad weather conditions. So holding inventory give some protection from unanticipated occurrences.

Finally the economic factor is associated with the economies of large size purchasing. Quantity discount, economies of transportation and lot size economies are some of the examples of economic factor.

As we have discussed above holding inventory has many advantages. However, it is also costly to hold inventory. So the question is how to mange inventory in a way that balances the cost and benefits of holding inventory. There are different inventory management approaches that have been in use to measure the efficient use of inventory in an organization. Below the classical cost optimization model is discussed.

2.1.3. Overview of classical cost minimization approach

The first and most known inventory model is EOQ model that optimizes the cost associated with inventory. It is also known as classical cost minimization model. The traditional inventory model is developed in 1915 by F.W. Harris and it is usually known as the Wilson formula, because R. H. Wilson published about this model on the Harvard business review in 19 34 and started to apply it extensively (Roach, 2005). The decision rule of EOQ model gives the quantity that should be ordered so as to minimize the total cost of inventory. The costs are inventory holding cost and ordering cost. Balancing the fixed cost per ordering with carrying costs is the basis to arrive the optimum order size. The model was formulated based on the following assumptions:

The demand rate is constant and continuous (the rate of depletion of inventory is constant). The order quantity need not be an integral number of units, and there are no minimum or maximum restrictions on size (there is no restriction on order size and storage capacity). The unit variable cost does not depend on the replenishment quantity, no shortages are allowed. As you see in the figure 2.1, on average the inventory is *Q/2* units in the store. The cycle time is $T = \frac{Q}{D}$ where *T* is time in year. Therefore, number of cycle (order) per year is $\frac{D}{Q}$

The total cost per year is given by,

$\boldsymbol{TC} = \boldsymbol{DC} + \frac{DA}{Q} + \frac{Qvr}{2}$ Where;

$\boldsymbol{TC}$ = total cost

$\boldsymbol{A}$ = ordering cost

$\boldsymbol{Q}$ = order size

$\boldsymbol{D}$ = annual demand

$\boldsymbol{v}$ = unit purchasing cost

$\boldsymbol{r}$ = carrying charge

The optimum order size is computed by differentiating the above total cost function with respect to *Q*

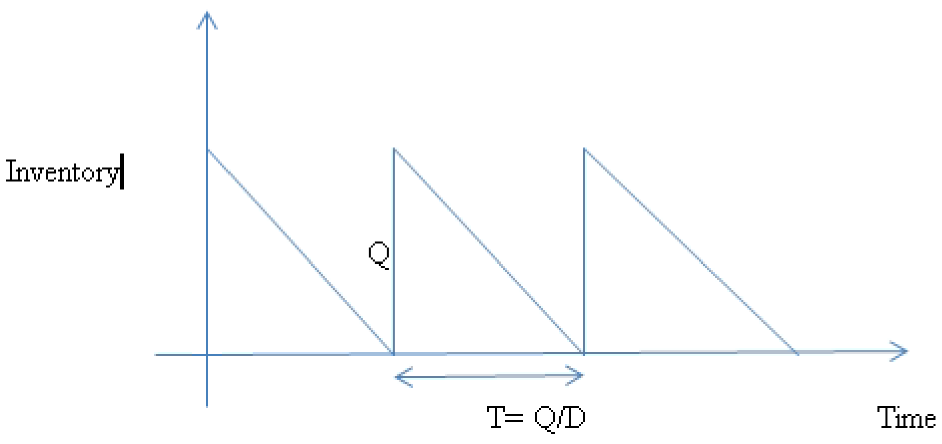

Figure 2.1: Inventory cycle of replenishment and consumption.

The classical inventory model excluded the possibility of stock out, but this is not always true. Not assuming the possibility of backordering (planned stock out) can be considered as one of the weakness of the traditional economic order quantity model. This model also disregards the possibility of capital constraint in the objective of cost minimization objective. Different researchers have tried to include the concept of back ordering to the objective of minimizing inventory cost. The next topic discussed some of the work of these researchers

2.2. Related literature review.

According to Gupta and Brennan (1992), back ordering may be unavoidable or necessary due to unanticipated demand surges, defaulting suppliers, perishability of products, high valued product in a volatile market, and space limitations. Limitation of storage space may also dictate back-ordering. For high price items and perishable products, backordering can be used as a hedging procedure to avoid too little or too high inventory (Gupta and Brennan 1992)

In markets like information technology (IT), consumer electronics and even in automobile, some customers show the willingness to wait until delivery (Che, Narasimhan and Padmanabhan, 2010). For example when you buy a new car the show room does not keep a stock of every variation in its models, but you choose the feature you want, the show room orders this from the manufacturer, and then you wait for the car to be delivered.

Nowadays including the concept of partial back ordering in inventory model is getting considerable attention by researchers. Many authors modeled partial back ordering using the concept of impatience. The concept of impatient was used by Hanssmann (1962) to model extreme shortage situations such as complete back order case and complete lost sale case. In (1996) Abad introduced the supposition that the fraction of back ordered demand is a function of the waiting time. The assumption was that customers who are willing to wait for back ordering during stock out period decreases with the length of waiting time.

Kim and Park (1985) studied a continuous review system with constant lead time where the fraction of the unfulfilled demand is back ordered and the back order cost is proportional to the length of time the back order existed.

Lee, Wu and Lei (2007) tried to develop an algorithm procedure for an inventory policy with back order discounts and variable lead time to find the optimum order quantity, optimum order cost, optimum lead time and the optimum back order price discount when the distributions of the lead time demand is mixture of normal distributions. These authors combine the work of Wu and Tsai (2001) and Pan and Hsiao (2001) and Ouyang and Chuang (2001). Ouyang and Chuang proposed that for most famous brands and fashionable commodities customers are willing to wait back-order if stock out occur. At the same time they stated that customers have the maximum patient to wait the back-order. This means that the size of backorder is dependent on the length of lead time. Hence they assumed back order rate is dependent on the length of lead time. In addition to this, Pan and Hsiao assumed that suppliers can offer a price discount on the stock out item in order to secure more back order; hence the back order rate is dependent on the back order price discounts offered by the supplier.

Gupta and Brennan (1992) developed two algorithms on lot sizing with back-orders. The first algorithm is a heuristic algorithm with a built-in “look-ahead” capability which seeks to minimize the total cost per period of set up, carrying and shortage costs. The “look ahead” continues until the total inventory cost per period rises. The second algorithm is the adaptation of the economic order quantity technique to accommodate back-order. This algorithm involves the calculation of both optimal lot size and the maximum permissible shortage quantity.

Sicilia, San-Jose, and Garcia-Laguna (2009) studied an optimal replenishment policy for an EOQ model with partial backlogging. The objective was to determine the economic lot size which optimizes the management of inventory system. Theses authors studied this inventory model based on the following assumptions. The item is a single product with independent demand. The planning horizon is infinite. The replenishment is instantaneous and the demand rate is known and constant. The order cost is fixed regardless of the lot size. The purchasing cost is known and constant. The holding cost is a linear function based on average inventory. The inventory is continuously revised. The model allows shortages, which are partially backlogged. The fraction of backlogged demand is a function of time the customer waits before receiving the goods and on time lapsed since the break in the stock took place. The cost of back order includes a fixed cost and a cost which is dependent on the length of time the back-order exists.

CHAPTER 3

3. FRAMEWORK

3.1. Planned back order for single item situation with no restriction on capital tied up in the inventory.

A backorder is demand that will be filled later than desired and a firm does not lose the sale when the inventory is depleted from shelf. The assumption is that customers are willing to wait until their order is delivered. However, backordering demand may be more costly than the routine order processing, since it often requires more expensive routing or higher prices for a shorter lead time from an alternative external source (Tersine 1994). For manufacturers it may necessitate overtime expenses.

“Allowing backorder means that some units are delivered in stock after they have been demanded” (Axsäter 2006, 59).

If back orders were very expensive, they would never be used. We will look at a situation where some stock is kept, but is not enough to cover all demand during the cycle time. In this case part of the demand is fulfilled from the stock and the remaining is back ordered. So the key question is how much of the demand should be taken from stock and how much from backorder? There is an intermediate range of backordering costs where it is optimal to incur some back orders towards the end of an inventory cycle.

Figure 1 shows the backordering inventory model. An order Q is placed when the stock on hand reaches the reorder point. Percentage of demand that will be backordered is x in the order cycle time. During the time period t_3 one order is placed. There is positive inventory during the time period t_1. The stock out period is t_2. The value of x increases as the stock out period (t_2) increases. The back ordering cost per unit per time (B) is expressed as a percentage of the unit purchasing price and is directly proportional to the length of the time delay (t_2). The assumption is that all stock out will be completely back-ordered. This means that there will not be any lost sales. The situation is illustrated in figure1 below.

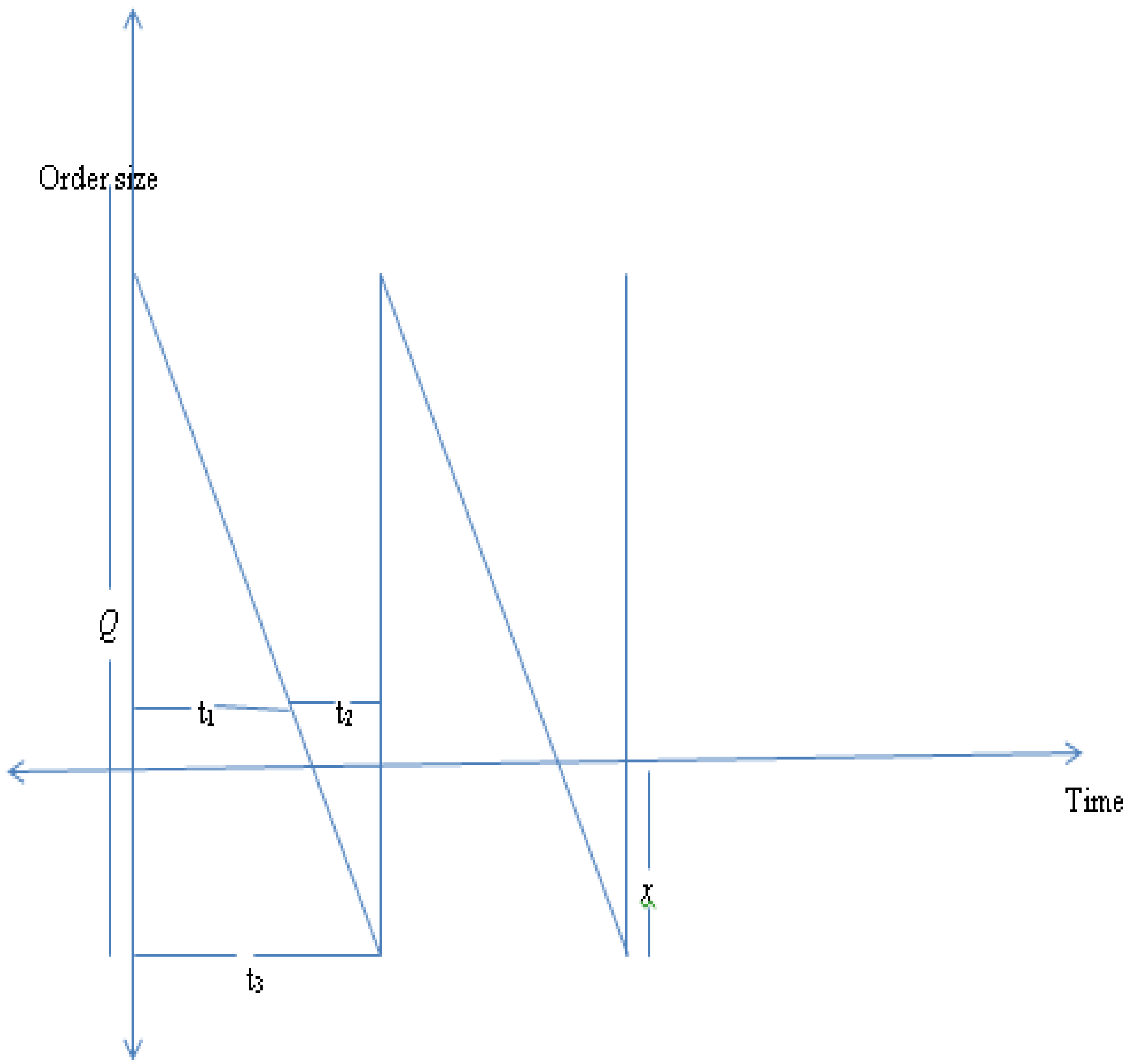

Figure1. Inventory model with planned backorder (source: Tersine 1994)

In order to formulate a mathematical model for this situation, we introduce the following notations for the single item case.

$\boldsymbol{Q}$ = order quantity

$\boldsymbol{x}$ = fraction (percentage) of demand that is backordered.

$(\mathbf{1}-\boldsymbol{x})$ = fraction of demand satisfied from stock

$\boldsymbol{B}$ = back ordering cost per unit per time expressed as percentage of purchasing price

$\boldsymbol{D}$ = Annual demand

A = Ordering cost

r = cost of capital tied up in inventory per dollar per year

As discussed by Axsäter (2006) the total logistics cost after backordering concept is included can be expressed as follows.

$$\boldsymbol{TRC(Q,x)} = \frac{DA}{Q} + \frac{Qvr(1-x)}{2}\left[\frac{(1-x)Q}{D}\ \frac{D}{Q}\right] + \frac{QxBv}{2}\left[\frac{Qx}{D}\ \frac{D}{Q}\right]$$

$$\boldsymbol{TRC(Q,x)} = \frac{DA}{Q} + \frac{Qvr(1-x)^2}{2} + \frac{Qx^2Bv}{2} \qquad (3.1)$$

Where;

$\boldsymbol{TRC(Q,x)} =$ Total Logistics cost

$\frac{D}{Q}$ = number of orders per year.

$\frac{DA}{Q}$ = Ordering cost

$\frac{Qvr(1-x)^2}{2}$ = holding cost

$\left[\frac{(1-x)Q}{D}\ \frac{D}{Q}\right]$ = number of time inventory is positive per year

$\left[\frac{Qx}{D}\ \frac{D}{Q}\right]$ = number of times inventory is negative per year

$\frac{Qx^2Bv}{2}$ = Back-ordering cost

Performing two partial derivations with respect to Q and x give us the optimal order quantity and percentage of back order. The detailed of this derivation can be found in Axsäter(2006)

$$x_B^* = \frac{r}{r+B} \qquad (3.2)$$

$$Q_B^* = \sqrt{\frac{2DA(r+B)}{vrB}}$$

$$Q_B^{*} = Q_{HW}\sqrt{\frac{r+B}{B}} \qquad (3.3)$$

You can see that the optimum order quantity when there is planned back-order is greater than the traditional optimum order quantity called Wilson formula (Q_{HW}). Given the optimum order size (3.3) and optimum proportion of back-order (3.2), you can compute the different cost elements such as ordering cost, holding cost and cost of back-ordering.

In the total relevant logistics cost function, the total ordering cost for family of items is given by $\frac{DA}{Q}$

Let TOC be ordering cost

At Q_B^{*} the total ordering cost is computed below.

$$TOC\,(Q_{Bi}^{*}) = \left(\frac{DA}{\sqrt{\frac{2DA\,(r+B)}{vrB}}}\right)$$

$$TOC\,(Q_B^{*}) = \sqrt{\frac{DArBv}{2(r+B)}}$$

$$TOC\,(Q_B^{*}) = \frac{1}{2}\sqrt{\frac{2DArBv}{(r+B)}}$$

$$TOC\,(Q_B^{*}) = \frac{\sqrt{2DArv}\,\sqrt{\frac{B}{(r+B)}}}{2} \qquad (3.4)$$

You can compare this result with the Wilson formula for ordering cost at the optimum. At the optimum order quantity of the Wilson formula, the total ordering cost OC($_{HW}$) is given by

$$OC_{HW} = \frac{1}{2}\sqrt{2DArv} \qquad (3.5)$$

Now you can compare the total ordering cost TOC and OC_{HW}

$$TOC\,(Q_B^{*}) = OC_{HW}\sqrt{\frac{B}{(r+B)}} \qquad (3.6)$$

The result shows that ordering cost with planned stock out is less than the ordering cost of the classical cost minimization approach. The number of orders per year, in the planned back-order model, is less than the number of orders in the classical cost minimization model. The size of order in this model, that satisfies the back ordered demand from the previous cycle time and part of the demand of the next cycle, is larger than that of classical cost minimization model. This results in fewer numbers of orders per year and lower ordering cost than the classical approach.

Total holding cost can also be computed using the same procedure. In the total relevant cost function holding cost is given by $\frac{Qvr(1-x)^2}{2}$. Let THC be total holding cost for a family of items. Total holding cost at $Q_B{}^*$ is computed below.

$$THC = \frac{Qvr\left(1-\frac{r}{r+B}\right)^2}{2}$$

$$THC = \frac{\sqrt{\frac{2DA(r+B)}{vrB}} \quad vr\left(1-\frac{r}{r+B}\right)^2}{2}$$

$$THC\ (Q_B{}^*) = \frac{B}{r+B}\sqrt{\frac{DAvrB}{2(r+B)}} \qquad (3.7)$$

This is the same as $\frac{B}{r+B}$ $TOC\ (Q^*)$.

This shows that holding cost is less than ordering cost, which is not true in the traditional cost minimization approach where total ordering cost is equals with total holding cost. Of course in the stock out period there is no holding cost incurred, rather it is backordering cost. This is the reason why holding cost is smaller than the ordering cost at the optimum.

Further more it is possible to compare the holding cost under planned stock out with the holding cost in the Wilson model.

$$THC\ (Q_B{}^*) = \frac{B}{r+B}\frac{1}{2}\sqrt{2DAvr} \quad \sqrt{\frac{B}{(r+B)}}$$

The total holding cost (HC_{HW}) under Harris Wilson model at optimum order quantity is given below.

$$HC_W = \frac{1}{2}\sum_1^n \sqrt{2DAvr} \qquad (3.8)$$

The relationship between total holding cost under planned stock out and Harris Wilson is as shown below.

$$THC\ (Q_B{}^*) = \frac{1}{2}\sum_1^n \sqrt{2DAvr}\ \sqrt{\frac{B}{(r+B)}} \quad \frac{B}{r+B}$$

$$\boldsymbol{THC}\,(Q_B{}^{*}) \;=\; HC_{HW}\;\sqrt{\frac{B}{(r+B)}}\;\;\frac{B}{r+B} \qquad (3.9)$$

The above relationship shows that holding cost under Harris Wilson model is higher than the planned stock out model.

$\boldsymbol{THC} \leq \;\; HC_{HW}$

In the classical approach when inventory reaches zero, the new order arrive immediately and shortage cannot happen. This means holding cost is incurred during the whole order cycle. In the planned back-order model formulation of this paper the researcher stated that the order cycle (t_3) is composed of t_1 and t_2. The t_3 is the order cycle. Inventory is positive during the time period t_1. The stock out period is t_2 and inventory is negative during this time. This means holding cost is only incurred for the time period t_1, while in the case of classical approach holding cost is incurred for the time period t_3. Since t_3 is longer than t_1, it is logical to have higher holding cost in the classical approach than holding cost in planned stock out approach.. We can also compute a formula to find cost of backordering for a family of items at optimum order size ($\mathbf{Q_B}{}^{*}$). Let *TBC* represent total backordering cost. The total back ordering cost is originally given by $\frac{Qx^2\, B\, v}{2}$. At optimum order size ($\mathbf{Q_B}{}^{*}$) total back ordering cost is given by,

$$TRC = \frac{Q_B{}^{*} x^2 B\, v}{2}. \qquad\qquad TBC\,(Q_B{}^{*}) \;=\; \frac{\sqrt{\dfrac{2DA}{vr\,(\frac{B}{r+B})}}\,(\frac{r}{r+B})^2\, B\, v}{2}$$

$$TBC\,(Q_B{}^{*}) = \frac{r}{r+B}\sqrt{\frac{DArBv}{2(r+B)}} \qquad (3.10)$$

$$TBC\,(Q_B{}^{*}) \;=\; x^{*}\; TOC\,(Q_B{}^{*}).$$

This is the same as the product of optimum percentage of back order (x^{*}) **and** *TOC* ($Q_B{}^{*}$). We can see that total ordering cost is higher than both back-ordering cost and holding cost. The other interesting result at optimum order size and optimum proportion of back-order is that the sum of holding cost and back-ordering cost is equal to the total ordering cost.

TOC = THC + TBC

$$\sqrt{\frac{DArBv}{2(r+B)}} = \frac{B}{r+B}\sqrt{\frac{DArBv}{2(r+B)}} \;+\; \frac{r}{r+B}\sqrt{\frac{DArBv}{2(r+B)}}$$

Finally, the total relevant cost (*TRC*) at optimum is given as the sum of ordering, holding and back ordering costs.

$TRC = TOC + THC + TBC$

$$TRC(Q^*) = \sqrt{\frac{DArBv}{2(r+B)}} + \frac{B}{r+B}\sqrt{\frac{DArBv}{2(r+B)}} + \frac{r}{r+B}\sqrt{\frac{DArBv}{2(r+B)}}$$

$$TRC(Q_B^*) = 2\sqrt{\frac{DArBv}{2(r+B)}}. \qquad (3.11)$$

At optimum order size the total relevant logistics cost is twice the amount of ordering cost. With Harris Wilson model, the total relevant Logistics cost at optimum order size is computed below. It is the sum of ordering cost and holding cost.

$TRC_{HW} = OC_{HW} + HC_{HW}$

$$TRC_{HW} = \sqrt{2DArv} \qquad (3.12)$$

To compare this with the total ordering cost under planned stock out, it is possible to rewrite $TRC(Q_B^*)$ as shown below.

$$TRC(Q_i^*) = \sqrt{2DArv}\sqrt{\frac{B}{(r+B)}}$$

$$TRC(Q_i^*) = TRC_{HW}\sqrt{\frac{B}{(r+B)}} \qquad (3.13)$$

Hence, the total logistics cost in the planned stock out model is less than the total logistics cost of Harries Wilson model.

What is discussed so far is a single item situation. What if we have a family of ‘n’ items that uses common resource like capital tied up in inventory and storage space and put restriction of these resources on the objective of minimizing total logistics cost. The main objective of this research is to extend the single item case discussed so far to a family of ‘n’ items that share the available capital resource where there is restriction on the maximum availability of this resource.

3.2. Planned back order for multi-item situation and restriction on capital tied up in the inventory.

The model has the objective of minimizing total relevant cost for family of items. It consider continuous review inventory model with complete back-ordering. The demand for the family of items is independent of each other. Further more the model has budget limitation as a constraint to the objective of minimizing the cost function. It relaxes the assumption of classical optimization model by introducing planned stock out.

The following assumptions and notations are made in developing the model.

(a) The demand rate of each item is constant.

(b) The replenishment of quantity of an item need not be an integral number of units.

(c) unit cost, ordering cost and inventory carrying cost are known and constant

(d) During the stock out period inventory is negative and all stock outs will be completely back-ordered

(e) The inventory is reviewed on continuous bases

(f) There is no quantity discount (the unit variable cost of the items does not depend on the quantity purchased)

(g) The entire order is delivered at the same time

(h) There is restriction on the availability of capital tied up in inventory. Inventory requires large amount of capital and capital is scarce resource. In the classical cost minimization approach there is no capital restriction and it is possible to purchase any order size with out any maximum limit. In reality it is usually difficult to purchase very big order size because of lack of availability of capital. The assumption is there only given maximum amount of money allocated to purchase inventory. The maximum available capital (money) is denoted by the letter *C*.

The objective is to determine the economic order size and percentage of back-order for family of items which optimizes the total inventory cost given that there is capital constraint. Considering the above assumptions two objective functions (models) are developed.

3.2.1. Model 1.

The first model is developed for a situation where the optimum order size in unconstrained optimization problem doesn’t require a capital that exceeds the maximum available amount.

In this case you disregard the capital constraint and try to minimize the total relevant cost. It is represented by the function given below in (3.1)

$$\textbf{Minimizing}\;\; TRC(Q,x) = \sum_{i=1}^{n} \frac{D_i A_i}{Q_i} + \frac{Q_i v_i r_i (1-x_i)^2}{2} + \frac{Q_i x_i^2 B_i v_i}{2}$$

3.2.2. Model 2.

The second model represents a situation where the unconstrained optimum order size cannot be purchased because it requires a capital that exceeds the maximum available capital. The objective is the same with the objective of model 1, but here there is capital constraint. This market it a constrained objective function and it is given below.

$$\textbf{Minimizing}\;\; TRC(Q,x) = \sum_{i=1}^{n} \frac{D_i A_i}{Q_i} + \frac{Q_i v_i r_i (1-x_i)^2}{2} + \frac{Q_i x_i^2 B_i v_i}{2}$$

$$\textbf{subject to} = \frac{1}{2}\sum_{i=1}^{n} Q_i v_i \leq C \qquad (3.14)$$

Where;

D_i = Demand for item i

Q_i = Order quantity for item i

$(1 - x_i)$ = percentages of back-order for item i

A_i = ordering cost for item i

r_i = cost of capital tied up in inventory per dollar per year for item i

B_i = back ordering cost expressed as percentage of purchasing price for item i

v_i = unit cost for item i

x_i = fraction of demand that is backordered

$\frac{1}{2} Q_i v_i$ = average capital tied up in inventory

C = maximum amount of the average capital allowed to be tied up in the inventory

CHAPTER 4

4. ANALYSIS AND DISCUSSION OF MODEL ONE AND TWO

4.1. Determining optimal order size and percentage of back order of model 1.

If the total average capital tied up in inventory is less that the maximum amount of available capital, we can disregard the constraint and find the optimum order quantity and percentage of back-order only using the objective function. To check this we first take only the objective function give in model 1 and compute the value of *Q* and x that minimize the total relevant cost. Then we compute the value of $\frac{Q_i v_i}{2}$ at optimum *Q* and x and compare it with the maximum amount of the available capital (C). If $\frac{Q_i v_i}{2}$ is less than *C* you disregard the constraint and take only the objective function to find the optimum order size (Q_i^*) and percentage of backorder (x_i^*)

If $\frac{Q_i v_i}{2}$ is less than or equal to *C* at optimum percentage of backorder and order size, you can partially derivate the objective function with respect to x_i and Q_i and find the value of x_i and Q_i that minimize the total relevant cost function give in model 1.The partial derivative with respect to the total relevant cost function with respect to x_i is given below.

$$\frac{dTRC(Q_i, x_i)}{dx_i} = -Q_i v_i r_i (1 - x_i) + Q_i x_i B_i v_i = 0$$

$$Q_i v_i r = Q_i x_i B_i v_i + Q_i v_i r x_i$$

$$x_i^* = \frac{r_i}{r_i + B_i} \qquad i = 1, 2,,,,n \qquad (4.1)$$

Where,

$r_i + B_i > 0$

x_i^*= optimum percentage of order that will be back-order

The variable x_i^* is the fraction of Q_i that is back-ordered and 1-x_i^* is the fraction of demand that can be delivered directly from the shelf. This means that 1-x_i^* is the fill rate showing the level of customer service. P_2 is the fraction of demand satisfied directly from the shelf with out backorder and lost sales. So 1-x_i^* is the same as P_2.

The explicit formula given in (4.1) shows that x_i^* is dependent only on r_i and B_i. If r_i is constant, an increase in B_i decreases x_i^*. If B decrease, x_i^* decreases. The parameter B_i is the penalty cost for not delivering directly from the shelf. The higher the penalty cost of not satisfying customers' demand directly from the shelf, the lower will be the proportion of backorder. The lower the penalty cost the higher will be the proportion of backorder. When the proportion of backorder (x_i^*) increases, the proportion of Q_i that will be used to satisfy customers' demand directly from the shelf (1-x_i^*) decreases. If the proportion of back order decreases, the amount of stock held to satisfy the demand directly from the shelf will increase.

Given B_i constant you can also analyze the impact of r_i on the proportion of backorder. The variable r_i is the cost of having one dollar of the item tied up in inventory for a unit time interval. When the cost of having one dollar tied up in inventory is high, it would be expensive to carry large amount of inventory. We can see an interesting relationship between r_i and the proportion of backorder (x_i^*). If r_i increases, x_i^* also increases and if r_i decrease, x_i^* also decreases. This shows that interest rate has direct relationship with the proportion of Q_i that will be back ordered. This means as the value of r_i increase the amount of inventory hold in the stock decreases and the proportion of demand that will be satisfied from backorder increases. So keeping B_i constant r_i has a direct relationship with the proportion of backorder and indirect relationship with the amount of inventory hold in the stock.

As we have seen the service level (P_2) for each item is (1-x_i^*). This is the same as (1- $\frac{r_i}{r_i+B_i}$) which is equals to $\frac{B_i}{r_i+B_i}$. This shows that the service level of each item may be different depending on the value of r_i and B_i .What could happen if a manager of a company wants to have the same service level for all family of items? This means that all the items will have the same proportion of demand that will be back-ordered .It can be expressed as $x_1^* = x_2^* = x_3^* = ,,,,,, = x_n^*$. So to have the same service level, the value of r_i and B_i should be the same for all items, otherwise $\frac{B_i}{r_i+B_i} = \frac{B_1}{r_1+B_1}$ =, , , , , = $\frac{B_n}{r_n+B_n}$

What is discussed so far is how to compute the optimum percentage of backorder. Now let us compute the order size that minimizes the total cost function (*TRC*). Given the result found in (4.1), it is possible to develop a formula that determines the optimum order size($Q_{Bi}{}^{*}$).The optimum order size can be found by taking the partial derivative of *TRC* with respect to Q_i and set it equals to zero.

$\frac{dTRC(Q_i, x_i)}{dQ_i} = -\frac{D_i A_i}{Q_i{}^2} + \frac{v_i r_i (1-x_i)^2}{2} + \frac{x_i{}^2 B_i\ v_i}{2} = 0$ and the second derivative must be positive to have a minimum point. The second derivative is $\frac{2D_i A_i}{Q_i{}^3}$ which is positive and hence we have a minimum point.

Solving the first derivatives equals to zero give the formula to find the optimum order quantity. The formula is shown below.

$$Q_{Bi}{}^{*} = \sqrt{\frac{2D_i A_i}{v_i r_i (1-x_i{}^{*})^2 + (x_i{}^{*})^2 B_i\ v_i}} \qquad (4.2)$$

$Q_{Bi}{}^{*}$ = optimal order quantity

This can be further simplified by substituting $\frac{r_i}{r_i + B_i}$ in the place of $x_i{}^{*}$ as shown below.

$$Q_{Bi}{}^{*} = \sqrt{\frac{2D_i A_i}{v_i r_i \left(1-\frac{r_i}{r_i+B_i}\right)^2 + \left(\frac{r_i}{r_i+B_i}\right)^2 B_i\ v_i}}$$

$$Q_{Bi}{}^{*} = \sqrt{\frac{2D_i A_i (r_i + B_i)}{v_i r_i B_i}} \qquad (4.3)$$

The result shows that the optimum order size is affected by the size of annual demand, ordering cost, purchasing price, carrying charge and cost of back ordering. Parameters $D_i, A_i,\ B_i$ and the variable r_i have positive relationship with the optimum order size. For example an increase in the amount of annual demand increases the optimum order size and a decrease in the annual demand decreases the optimum orders size. However, purchasing price has negative relationship with the optimum order quantity. An increase in the purchasing price decreases the order size and a decrease in the purchasing price decreases the optimum order size

4.2. Determining optimal order size and percentage of back order of model 2

First compute the optimum order size without considering the capital constraint, take the average capital tied up in inventory at the optimum order size and compare it with the available budget size. If the optimum order size require a capital exceeding the maximum budget allowed, it is not possible to order the optimum order size found with unconstrained optimization model. This is because there will not be enough money to purchase the optimum order size. This situation requires considering the capital constraint in the process of finding optimal order size and back-order. In this case, the cost minimization objective becomes an optimization problem subject to capital constraint. This situation is mathematically expressed in (3.14) of chapter three.

Minimizing $TRC(Q,x,) = \sum_{i=1}^{n} \frac{D_i A_i}{Q_i} + \frac{Q_i v_i r_i (1-x_i)^2}{2} + \frac{Q_i x_i^2 B_i v_i}{2}$

$$subject\ to: \frac{1}{2}\sum_{i=1}^{n} Q_i v_i \leq C$$

The maximum available capital is *C* and if the optimums order size of the unconstrained objective function requires a capital exceeding *C*, the best option we have in the short run is to utilize all the available capital. In this case, to utilize all the available capital, the average capital tied up in inventory must equal to *C* i.e. $\frac{1}{2}\sum_{i=1}^{n} Q_i v_i = C$. Then we have cost minimization objective function with capital constraint as given below.

$$TRC(Q_i, x_i) = \sum_{i=1}^{n} \frac{D_i A_i}{Q_i} + \frac{Q_i v_i r_i (1-x_i)^2}{2} + \frac{Q_i x_i^2 B_i v_i}{2}$$

$$\text{subject to:} \frac{1}{2}\sum_{i=1}^{n} Q_i v_i = C \qquad (4.4)$$

This constrained optimization is solved using the concept of Lagrange function. This first task in solving Lagrange function is to change constrained objective in to unconstrained objective function. The unconstrained objective function (Lagrange function) is shown below. The basic concepts of Lagrange function can be found in any standardize calculus books.

$$L(Q_i, x_i, \lambda) = \sum_{i=1}^{n} \left(\frac{D_i A_i}{Q_i} + \frac{Q_i v_i r_i (1-x_i)^2}{2} + \frac{Q_i x_i^2 B_i v_i}{2}\right) + \lambda\left(\frac{1}{2}\sum_{i=1}^{n} Q_i v_i - C\right). \qquad (4.5)$$

Then we find the partial derivatives of the Lagrange function with respect to x_i , Q_i and λ and set each partial derivatives equals to zero. You can find the detail in Appendix A. Setting the first partial derivative of the Lagrange function with respect to x_i equal to zero gives us the optimum percentage of order size that will be backordered $(x_{B_i})^*{}_{cap}$

$$\frac{dL(Q_i, x_i, \lambda)}{dx_i} = \frac{Q_i v_i r_i(-2+2x_i)}{2} + \frac{2x_i Q_i B_i v_i}{2} = 0$$

$$(x_{B_i})^*{}_{cap} = \frac{r_i}{r_i + B_i} \qquad (4.6)$$

The optimum percentage of back-order under constrained condition is the same with the optimum percentage of back order with out capital constraint given in (3.2). It is determined by the value of the parameter B_i and the variable r_i. The result shows that percentage of back order is independent of capital constraint.

Using the same procedure you can compute the partial derivative of the Lagrange function with respect to Q_i and set it equal to zero and find a formula that is used to find the optimum order quantity. Refer Appendix A to see the partial derivation and detail computation made to find the optimum order size $(Q_{B_i})^*{}_{cap}$ of the constrained optimization problem.

$$\frac{dL(Q_i, x_i, \lambda)}{d\ Q_i} = -\frac{D_i A_i}{Q_i^2} + \frac{v_i r_i(1-x_i)^2}{2} + \frac{x_i^2 B_i v_i}{2} + \frac{1}{2}\lambda v_i$$

The second derivative must be positive to have value of $(Q_{B_i})^*{}_{cap}$ that minimizes the objective function. The second derivative is of course positive and hence we can have minim point. You can see this in Appendix A. Setting the first derivative with respect to Q_i equal to zero gives us a formula to find the optimum order size that minimizes the total logistics cost. The formula is given below.

$$(Q_{B_i})^*{}_{cap} = \sqrt{\frac{2D_i A_i}{v_i r_i(1-x_i)^2 + x_i^2 B_i v_i + \lambda v_i}} \qquad (4.7)$$

The result shows that the variables v_i, r_i, x_i, λ and the parameters D_i , A_i and B_i affects the determinations of optimum order size. However, at optimum percentage of back order, we can replace the proportion of back order (x_i) by the variables determining it such as r_i and B_i. If we substitute x_i by $\frac{r_i}{r_i+B_i}$ we will have another expression as shown below.

$$(Q_{B_i})^*{}_{cap} = \sqrt{\frac{2D_iA_i}{v_ir_i\left(1-(\frac{r_i}{r_i+B_i})\right)^2+\left(\frac{r_i}{r_i+B_i}\right)^2 B_iv_i \ +\lambda v_i}}$$

$$(Q_{B_i})^*{}_{cap} = \sqrt{\frac{2D_iA_i}{\frac{v_iB_i\ r_i}{r_i+B_i}+\lambda v_i}}$$

$$(Q_{B_i})^*{}_{cap} = \sqrt{\frac{2D_iA_i}{v_i\ (\frac{r_i}{r_i+B_i}+\lambda)}} \qquad (4.8)$$

Considering change in one variable while other variables remain constant, you can see the relationship of each parameters and variables with optimal order size. Ordering cost and the size of annual demand has direct relationship with the size of order quantity. When the ordering cost increases, the order size also increases and when ordering cost decreases, the order size also decreases. The same is true for annual demand. The back ordering cost, purchasing cost, the carrying charge and the variable lambda (λ) have indirect relationship with the order size. An increase in any of these variables will decrease the order size and the reverse is true. For example when the value of v_i increases, the order size will decrease and when v_i decreases, order size increases.

In the original model formulation there was no variable called lambda. It is introduced in the process of solving the constrained optimization problem. This variable has its own economic interpretation. We will discuss its economic meaning later. With the same procedure used to find a closed formula for order size and percentage of back-order it is also possible to find a formula that is used to find the value of lambda at optimum point. We take the partial derivative of the Langrage function with respect to λ and set it equal to zero. The partial derivative is shown below

$\frac{dL(Q_i, x_i\, \lambda)}{d\lambda} = \frac{1}{2}\sum_{i=1}^{n} Q_iv_i - C.$ = 0. At optimum order size this equation looks like the equation shown below.

$$\frac{1}{2}\sum_{=1}^{n}\sqrt{\frac{2D_iA_i\ v_i}{(\frac{B_ir_i}{r_i+B_i}\ +\lambda)}} = C.$$

The assumption so far was that back ordering cost and carrying charge can take different value for different items. This makes the process of finding a formula for lambda some what difficult. However, we are dealing with family of items and it may be possible to assume the same back ordering cost and carrying charge for all items. Here let us assume that back ordering cost and carrying charge are the same for all family of items. In this case the back ordering cost (B_i) and carrying charge (r_i) are assumed to be the same for all family of items. So r and *B* does not have indices and $\left(\frac{B_i r_i}{r_i+B_i} + \lambda\right)$ expression is changed to $\left(\frac{Br}{r+B} + \lambda\right)$.Then we have new expression as shown below. Refer Appendix *A* for further detail.

$$\sum_{i=1}^{n} \sqrt{\frac{2D_iA_iv_i}{\left(\frac{Br}{r+B}+\lambda\right)}} = 2C$$

Now it is possible to take $\left(\frac{Br}{r+B} + \lambda\right)$ out from the summation notation. Finally we have a formula that is used to compute the value of λ as shown below.

$$\lambda = \left(\frac{1}{2C}\sum_{i=1}^{n}\sqrt{2D_iA_iv_i}\right)^2 - \frac{Br}{r+B} \qquad (4.9)$$

One may be interested on the economic interpretation of the variable lambda. Lambda is also known as shadow price. The value of lambda measures the extra value that would come from increasing the available capital resource by one unit. It shows the amount of saving in total logistics cost by increasing the available capital by one unit.

In the formula representing the optimum order quantity (4.8) there is a variable called lambda (λ). It is possible to replace lambda in (4.8) by the expression in (4.9) and find new expression representing optimum order quantity as shown below.

$$(Q_{B_i})^*_{cap} = \sqrt{\frac{2D_iA_i}{v_i\left(\frac{Br}{r+B} + \left(\frac{1}{2C}\sum_{i=1}^{n}\sqrt{2D_iA_iv_i}\right)^2 - \frac{Br}{r+B}\right)}}$$

$$(Q_{B_i})^*_{cap} = 2C\sqrt{\frac{2D_iA_i}{v_i\left(\sum_{i=1}^{n}\sqrt{2D_iA_iv_i}\right)^2}} . \qquad (4.10)$$

Where;

v_i and $\left(\sum_{i=1}^{n}\sqrt{2D_iA_iv_i}\right)^2$ are greater than zero.

This shows that optimum order size is determined by annual demand, the budget size, ordering cost and purchasing price. The variable *r* and the parameter *B* are not part of this formula showing that carrying charge and back ordering cost have no effect on determining the optimum order size. This is not true in Harris Wilson and unconstrained planned stock out models. In these models *r* and *B* affect the determination of optimum order size.

The formula to compute $(Q_{B_i})^*{}_{cap}$ can be further manipulated and simplified as shown below.

$$(Q_{B_i})^*{}_{cap} = \frac{2C}{\sum_{i=1}^{n}\sqrt{2D_iA_iv_i}}\sqrt{\frac{2D_iA_i}{v_i}} = \frac{2C\sqrt{r}}{\sum_{i=1}^{n}\sqrt{2D_iA_iv_i}}\sqrt{\frac{2D_iA_i}{v_ir}}.$$

This is the same as $\frac{2Cr}{\sum_{i=1}^{n}\sqrt{2D_iA_iv_ir}}(Q_{HW})_i$

$$(Q_{B_i})^*{}_{cap} = \frac{2Cr}{\sum_{i=1}^{n}(TRC_{HW})_i}(Q_{HW})_i$$

You can see the relationship of $(Q_{B_i})^*{}_{cap}$ with the Wilson's order size and the summation of Wilson's total relevant cost of family of items. This shows that knowing Wilson's order quantity and total relevant cost of family of items, budget size and carrying charge is enough to compute the capital constrained optimum order size. This can make computation of the optimum order size much simpler. It needs only to compute the Wilson's optimum orders sizes and its total cost for family of items. The value of the parameter *C* and the variable *r* are given. The budget size, carrying charge and Wilson order size have a direct relationship with the optimum order size of planned back order with capital constraint for family of items. The sum of the cost of family of items has indirect relationship with the optimum order size of planned back order with capital constraint for family of items.

CHAPTER 5

5. NUMERICAL EXAMPLE

A family of six different items is considered. The ordering cost (A) is the same for all family of items, back-ordering cost (B) is also assumed to be the same for all items. The purchasing price (v_i) and annual demand (D_i) of each items is different. The maximum available budget for the purchase of inventory is 30,000 dollar. Ordering cost is 200, back-ordering cost (B) is 0.10. The carrying charge is 0.20 and is the same for all family of items. The demand and purchasing price of each items is given in the table below.

Table5.1.Annual demand and purchasing price of each items

Items	Annual demand(D_i)	Purchasing price(v_i)
1	500	25
2	350	150
3	400	130
4	800	50
5	470	80
6	620	75

In the table below the optimum order size and optimum percentage of back order for each of the six items is computed using the planned back-ordering model. It also used to compare the difference among the classical Wilson's model, planned back order with out capital constraint and planned back-order with capital constraint. The result of order size is approximated to the nearest whole number.

Table5.2. Comparing the optimum order size under three different models - classical planned back order and planned back order with capital constraint models.

Items	$(Q_{HW})_i = \sqrt{\frac{2D_iA_i}{v_ir}}$	$(Q_{B_i})^* = \sqrt{\frac{2D_iA_i(r_i+B_i)}{v_ir_iB_i}}$	$(Q_{B_i})^*_{cap} = \frac{2Cr}{\sum_{i=1}^{n}(TRC_{HW})_i}(Q_{HW})_i$
1	200	346	228
2	68	118	77
3	78	136	89
4	179	310	204
5	108	188	123
6	129	223	147

Note that $\sum_{i=1}^{n}(TRC_{HW})_i$ = 10540.94 dollar. The detailed calculation of the total relevant cost is given in appendix B. The result in the above table shows that the order sizes computed using classical cost minimization model $(Q_{HW})_i$ is lower than both the constrained and unconstrained planned stock out models. It also shows that $(Q_{B_i})^*$ is higher than $(Q_{B_i})^*_{cap}$. However, the percentage of back- order is the same in both constrained and unconstrained models. It is 0.66.7. This means that 66.7% of items ordered are fulfilled from back order. Only 33.3% (1- 66.7%) of the customers order is fulfilled directly from the shelf. This is the same as customers service level (P_2); percentage of customers order fulfilled directly from the shelf which is 33.3%. The average capita tied up in inventory using Wilson formula, planned back-order without constraint and planned back-order with constraint are also computed in the table below. This helps to see if the available capital is enough to purchase the optimum order size of unconstrained planned stock out model.

Table5.3. Average capital tied up in inventory under Wilson formula, planned back-order without constraint and planned back-order with constraint

Items	$0.5(Q_{HW})_i v_i$	$0.5\,(Q_{B_i})^* v_i$	$0.5(Q_{B_i})^*{}_{cap} v_i$
1	2500	4325	2850
2	5100	8850	5775
3	5070	8840	5785
4	4475	7750	5100
5	4320	7520	4920
6	4837.5	8362.5	5512.5
Total average capital tied up in inventory	$\sum_{1}^{6} 0.5(Q_{HW})_i v_i$ $= 26302.5$	$\sum_{1}^{6} 0.5\,(Q_{B_i})^* v_i$ = 45647.5	$\sum_{1}^{6} 0.5\,(Q_{B_{i}cap})^* v_i$ $= 29942.5$

The total average capitals tied up in inventory in the three models are 26302.5, 45647.5 and 29942.5 dollar respectively. The average capital tied up in inventory is higher than the available budget size in the unconstrained planned stock out model. However, the average capital tied up in inventory in the planned stock out with capital constraint is with in the budget constraint. Since the order size is approximated to the nearest whole number, the value of the total capital tied up in inventory may not be exactly the same as the available budget size (C) i.e. why in the constrained model the average capital tied up in inventory is not exactly equals to 30,000 dollar (the available budget).

What we have discussed so far is the optimum order size, the optimum percentage of backorder and the average capital tied up in inventory of the three models. It is also possible to calculate the total cost of the three models. Especially by observing the total cost of all family of items and capital tied up in inventory of the constrained and unconstrained planned stock out model, it is possible to draw important decision implication.

Table 5.4.Comparison of the total cost and capital tied up in the three different models.

	Harris Wilson model	Planned stock out model	Planned stock out with capital restriction model
Total cost(TRC)	10540.94	6085.8	6854.9
Capital tied up in inventory	26302.5	45647.5	29942.5

The total cost for capital constrained stock out model at optimum order size and optimum percentage of back order is 6854.9 dollar. The total cost for unconstrained stock out model at optimum is 6085.8dollar. The difference in cost between constrained and unconstrained stock out model is 769.1 dollar. The total capital tied up in inventory for planned stock out model and planned stock out with out constraint model are 45647.5 and 29942.5 dollar respectively. The difference is -15705.

Let Δ_{Cost} be the difference between the total cost of constrained planned stock out model and unconstrained planned model. And let Δ_{Cap} be the difference of capital tied up in inventory between Constrained and unconstrained stock out models. It may be interesting to see the ratio of Δ_{Cost} to Δ_{Cap}

$\frac{\Delta Cost}{\Delta Cap} = \frac{769.1}{-15705}$ = -0.049. This ration has managerial implication. It tells the manager that there is a possibility of reducing the total logistics cost by increasing the available budget the budget size to size to 45647.5, results in a reduction on the total logistics cost on average by 0.049. This means that raising an additional 15705 dollars would bring a cost reduction of 769.6 (0.049 * 15705). It tells us the average cost saving of one additional dollar investment if the budget size increases to 45647.5. The decision to raise the budget size to 45647.5 depends on the cost of raising capital. If the average cost of raising the additional capital 15705 (45647.5 – 29942.5) is less than 0.049, the manager can increase the budget size and get the benefit of cost reduction. However, if the average cost of raising 15705 dollar is more that 0.049, the manager shall not increase the budget size to 45647.5. This is because on average the benefit of increasing the budget size is 0.049, while the cost of raising 15705 is greater than 0.049.The net benefit is negative and increasing the budget size by such magnitude cannot add value to the organization.

However, the ration (0.049) doesn't tell us the cost reduction caused by increasing the budget size only by one dollar. The exact value addition (cost reduction) derived from rising the capital constraint by one dollar is explained by the value of lambda. This is the same as the concept of shadow price. It measures the extra value that would be added by increasing the available capital resource by one unit. We have seen that the meaning of lambda is related with the concept of shadow price in the previous chapter. So in this example lambda tells us the benefit of cost reduction by increasing the budget size from 30,000 to 30,001.We can use the formula for lambda and make cost benefit analysis of relaxing the budget constraint by one dollar. From this numerical example it is possible to compute the value of lambda (λ).

$$\lambda = \left(\frac{1}{2C}\sum_{i=1}^{n}\sqrt{2D_iA_iv_i}\right)^2 - \frac{Br}{r+B}$$

$$\lambda = \left(\frac{23570.26}{60000}\right)^2 - \frac{(0.2)0.1}{(0.2+0.1)} = 0.09317$$

It shows the amount of saving in the total logistics cost by increasing the available capital by one unit. If the budget size increases by one dollar, the total logistics cost would decrease by 0.09317 dollar. It is the additional value generated when the constraint is relaxed by one unit. It is an important decision variable when management want to relax the constraint. It shows the maximum amount that a company is willing to pay to increase the budget size by one dollar. If the cost of acquiring the first one dollar is less than 0.09317, it is beneficial to increase the budget size. For example if the cost of acquiring one dollar is 0.099 dollar, the management will not be willing to increase the budget size because the benefit (cost saving) associated with increasing the budget size by one dollar is 0.09317 which is less than the cost of getting one dollar (0.099)

CHAPTER 6

6. SUMMARY

Inventory represents a significant portion of investment in assets for many companies. In addition to requiring significant investment, there are costs associated with acquiring and holding inventory such as purchasing cost, holding cost, ordering cost and stock out costs. One of the key inventory management decisions is determining optimum order size that minimizes the total costs associated with acquiring and holding inventory given that there is limitation on the availability of capital to invest in inventory. The well known approach used to find the optimum order size the classical cost minimization approach. It is also known as Harris Wilson model. According to this approach there is no possibility for stock out and no shortage of capital to purchases any inventory size. This approach assumes holding and ordering cost as the only relevant inventory costs. This approach try to determine the optimum order size by minimizing the total inventory holding costs and ordering costs.

This master thesis has modified some of the assumptions of the classical cost minimization approach and derived a formula used to find optimum order size. The researcher assumed that there is a planned stock out situation and all the stock out is completely back ordered and hence there is planned back ordering cost. The concept of capital concept of capital constraint is also introduced in the research model and hence there is given amount of money available to invest on inventory. The objective on this research is to find a closed formula to find optimum order size and optimum back order (expressed as percentage of order size) that minimizes the total inventory costs. The results of this research are summarized as follows;

- The formula to find optimum percentage of back order (x_i^*) is given by $\frac{r_i}{r_i+B_i}$. This shows that optimum percentage of back order is independent of annual demand, ordering cost and purchasing price.
- The optimum percentage of back order is independent of capital constraint. It is the same in the planned stock out models with out and with capital constraint.
- The formula to find the optimum order size(Q_{Bi}^*) in the planned stock out with out capital constraint is derived and is given by $Q_{Bi}^* = \sqrt{\frac{2D_iA_i(r_i+B_i)}{v_ir_iB_i}}$
- The formula to find the optimum order size ($(Q_{B_i})^*cap$) in the planned stock out with capital constraint is derived and is given

by $(Q_{B_i})^{*}cap \; = 2C\sqrt{\frac{2D_iA_i}{v_i\;\left(\sum_{i=1}^{n}\sqrt{2D_iA_iv_i}\right)^{2}}}$. This is the same as $\frac{2Cr}{\sum_{i=1}^{n}\sqrt{2D_iA_iv_ir}}$

$(Q_{HW})_i$.

- The result of this research also shows that optimum order size in the classical cost minimization approach is less than the capital constrained planned back order model
- In a constrained optimization problem the question of relaxing the constraint resource is an important management decision. This is related with the concept of shadow price which is explained by the value of lambda (λ) in this research. The formula to compute the value of lambda is also derived and is given as $\lambda = \left(\frac{1}{2C}\sum_{i=1}^{n}\sqrt{2D_iA_iv_i}\right)^{2} - \frac{Br}{r+B}$. It tells us the benefit (value addition) of increasing the available capital by one dollar. Management can compare this result with the cost of raising one unit of additional capital and make the whether or no to raise capital

Finally there are possibilities for further research. It is possible to modify the complete back ordering assumption in this research to a stock out situation where part of the stock out is back ordered and the remaining is lost sales. It is also possible to research on constrained cost minimization approach based on stochastic demand.

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APPENDICES

A. Minimizing total relevant cost subject to capital constraint.

$$TRC(Q,x,) = \sum_{i=1}^{n} \frac{D_i A_i}{Q_i} + \frac{Q_i v_i r_i (1-x_i)^2}{2} + \frac{Q_i {x_i}^2 B_i v_i}{2} \quad (1)$$

$$subject\ to = \frac{1}{2}\sum_{i=1}^{n} Q_i v_i = \mathrm{C} \quad (2)$$

This constrained optimization is solved using the concept of Lagrange function. The Lagrange function is as shown below.

$$L(Q,x,\lambda) = \sum_{i=1}^{n} \frac{D_i A_i}{Q_i} + \frac{Q_i v_i r_i (1-x_i)^2}{2} + \frac{Q_i {x_i}^2 B_i v_i}{2} + \lambda\left(\frac{1}{2}\sum_{i=1}^{n} Q_i v_i - C\right). \quad (3)$$

Then we find the first order derivatives with respect to x_i , Q_i and λ

$$L(Q_i,x,\lambda) = \sum_{i=1}^{n} \frac{D_i A_i}{Q_i} + \frac{Q_i v_i r_i (1-x_i)^2}{2} + \frac{Q_i {x_i}^2 B_i v_i}{2} + \lambda\left(\frac{1}{2}\sum_{i=1}^{n} Q_i v_i - C\right). \quad (4)$$

Then we find the first order derivatives with respect to x_i , Q_i and λ

$$\frac{dL(Q_i,x_i\,\lambda)}{dx_i} = \frac{Q_i v_i r_i(-2+2x_i)}{2} + \frac{2x_i Q_i B_i v_i}{2} \quad (5)$$

$$\frac{dL(Q_i,x_i\,,\ \lambda)}{dQ_i} = -\frac{D_i A_i}{Q_i^2} + \frac{v_i r_i (1-x_i)^2}{2} + \frac{{x_i}^2 B_i v_i}{2} + \frac{1}{2}\lambda v_i \quad (6)$$

$$\frac{1}{2}\sum_{i=1}^{n} Q_i v_i - C. \quad (7)$$

Setting the partial derivative in (5) gives us the optimum percentage of back-order (x_i^*)

$$\frac{Q_i v_i r_i(-2+2x_i)}{2} + \frac{2x_i Q_i B_i v_i}{2} = 0$$

$$r_i x + B_i x - r_i = 0$$

$\frac{d2L(Q_i,x_i\,\lambda)}{d2\,x_i} = r_i + B_i$. The second derivative should be positive so that we have a minimum point. The second derivative is of course positive and hence, we have minimum point. So setting the fist derivative to zero gives us the formula given below which is used to calculate optimum percentage of

$$x_i^* = \frac{r_i}{r_i + B_i} \quad (8)$$

Where,

x_i^* = Optimum percentage of backorder

B_i = cost of back ordering

r_i = carrying charge

Setting the partial derivatives of (6) gives you the optimum order size.

$$\frac{dL(Q_i, x_i, \lambda)}{dQ_i} = -\frac{D_i A_i}{Q_i^2} + \frac{v_i r_i (1-x_i)^2}{2} + \frac{x_i^2 B_i v_i}{2} + \frac{1}{2}\lambda v_i = 0$$

$\frac{d2L(Q_i, x_i, \lambda)}{d2Q_i} = \frac{2D_i A_i}{Q_i^2}$ which is positive and hence we have minimum point.

$$(Q_{B_i})^* cap = \sqrt{\frac{2D_i A_i}{v_i r_i (1-x_i)^2 + x_i^2 B_i v_i + \lambda v_i}}$$

If you substitute x_i by $\frac{r_i}{r_i + B_i}$ you have another expression.

$$(Q_{B_i})^* cap = \sqrt{\frac{2D_i A_i}{\frac{v_i B_i r_i}{r_i + B_i} + \lambda v_i}}$$

$$(Q_{B_i})^* cap = \sqrt{\frac{2D_i A_i}{v_i \left(\frac{B_i r_i}{r_i + B_i} + \lambda\right)}} \qquad (9)$$

At the optimum order size $(Q_{B_i})^*_{cap}$ you can develop a formula to compute the value of λ. Using (7) and (9) we can have the mathematical expression for the variable lambda. From (7) we have $\frac{1}{2}\sum_{i=1}^{n} Q_i v_i = C$. Then you substitute $(Q_{B_i})^*_{cap}$ by the right hand side expression of equation (9) and you get the expression to find the value of λ

$$\frac{1}{2}\sum_{i=1}^{n} \sqrt{\frac{2D_i A_i}{\frac{v_i B_i r_i}{r_i + B_i} + \lambda v_i}}\, v_i = C$$

Here it is some what difficult to find a simple formula that express lambda. This is mainly because of the indices each variables and parameters have. Here I assumed cost of back ordering and carrying charge are the same for all family of items. In this case carrying charge and cost of back ordering does not have indices and possible to take out *r* and *B* from the summation notation. The formula for lambda is simplified and presented below.

$$\sum_{i=1}^{n} \sqrt{\frac{2D_i A_i v_i}{\left(\frac{Br}{r+B}+\lambda\right)}} = 2C$$

$$\sum_{i=1}^{n} \frac{\sqrt{2D_i A_i v_i}}{\sqrt{\left(\frac{Br}{r+B}+\lambda\right)}} = 2C$$

$$\frac{1}{2C}\sum_{i=1}^{n} \sqrt{2D_i A_i v_i} = \sqrt{\left(\frac{Br}{r+B}+\lambda\right)}$$

$$\left(\frac{1}{2C}\sum_{i=1}^{n} \sqrt{2D_i A_i v_i}\right)^2 = \frac{Br}{r+B}+\lambda$$

$$\lambda = \left(\frac{1}{2C}\sum_{i=1}^{n} \sqrt{2D_i A_i v_i}\right)^2 - \frac{Br}{r+B} \qquad (10)$$

We can rewrite the formula for optimum order quantity given at (9) by substituting $\left(\frac{1}{2C}\sum_{i=1}^{n} \sqrt{2D_i A_i v_i}\right)^2 - \frac{Br}{r+B}$ in the place of lambda as given below

$$(Q_{B_i})^{*}cap = \sqrt{\frac{2D_i A_i}{v_i\ \left(\frac{Br}{r+B}+\lambda\right)}}$$

$$(Q_{B_i})^{*}cap = \sqrt{\frac{2D_i A_i}{v_i\ \left(\frac{Br}{r+B}+\left(\frac{1}{2C}\sum_{i=1}^{n}\sqrt{2D_i A_i v_i}\right)^2 - \frac{Br}{r+B}\right)}}$$

$$(Q_{B_i})^{*}cap = 2C\sqrt{\frac{2D_i A_i}{v_i\ \left(\sum_{i=1}^{n}\sqrt{2D_i A_i v_i}\right)^2}} \qquad (11)$$

Where;

C, v_i, and $\left(\sum_{i=1}^{n}\sqrt{2D_i A_i v_i}\right)^2 > 0$

$(Q_{B_i})^{*}{}_{cap}$ = optimum order quantity

C = the maximum available capital (budget size)

v_i = purchasing cost

D_i = Annual demand. A_i = ordering cost

B. Comparison of total relevant cost under Harris Wilson, Planned stock out with and with out capital constraint.

The total relevant cost for $TRC\ (Q_{B_i})^*$ and TRC$(Q_{B_i})^*{}_{cap}$ can be computed using the formula given below.

$$\sum_{i=1}^{n} \frac{D_i A_i}{Q_i} + \frac{Q_i v_i r_i r(1-x_i)^2}{2} + \frac{Q_i x_i^2 B_i v_i}{2}$$

The total relevant cost of the family of six items can be computed using the formula

$$\sum_{i=1}^{n} \sqrt{2 D_i A_i v_i}$$

Item	$TRC(Q_{HW})_i$	$TRC\ (Q_{B_i})^*$	TRC$(Q_{B_i})^*{}_{cap}$
1	1000	577.3	848.6
2	2049.4	1182.8	1294.1
3	2039.6	1177.7	1284.6
4	1788.9	1032.8	1124.3
5	1734.4	1001.5	1092.2
6	1928.7	1113.7	1211.1
Total cost	10540.9	6085.8	6854.9

$\sum_1^6 TRC(Q_{HW})_i$ = 10540.9

$\sum_1^6 TRC\ (Q_{B_i})^*$ = 6085.8

$\sum_1^6 TRC(Q_{B_i})^*{}_{cap}$ = 6854.9