Master's degree thesis

LOG950 Logistics

Introducing profit maximization in inventory routing problems

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Preface

The topic of this Master's thesis is "Introducing profit maximization in inventory routing problems".

The thesis was written according to the requirements for the Master of Science in Logistics degree. The thesis was written at Molde University College – Specialized University in Logistics. A part of the thesis was written at Federal University of Minas Gerais during a three-week stay in Brazil supported by the project UTF-2016-short-term/10123 with the title "Coordinated Optimization of Ports and Ships".

The work was supervised by Professor of Quantitative Logistics of Molde University College (Norway) Lars Magnus Hvattum and Professor of Computer Science Department of Federal University of Minas Gerais (Brazil) Sebastián Alberto Urrutia.

The development of two models of IRP with profit maximization for two types of market (monopoly and perfect competition), the linearization of the models, the experiments on a set of randomly generated instances and the analysis of the results have been performed by the author.

Summary

In this paper inventory routing problem (IRP) is considered. A basic IRP is concerned with the distribution of a single type of product from a single facility to a set of customers with given demand and inventory capacities over a given planning horizon. The problem is to determine for each discrete time period the quantity to deliver to each customer and the vehicle routes. The objective of the IRP is minimization of the sum of inventory and transportation costs without causing stockouts at any of the customers. However, in a supply chain context, where managers try to increase companies' profitability, the focus of planning decisions in such an integrated problem as an IRP should be on profit maximization. Ways of profit maximization depend on the type of the market, where a company operates: monopoly or perfect competition.

In this master's degree thesis profit maximization was introduced in inventory routing problems. The literature overview of existing inventory routing problems with profit maximization was provided. Two models of IRP with profit maximization for monopoly and perfect competition were developed. The model for monopoly allows to set the prices finding the optimal trade-off between volume and margin according to the demand function. The model for a perfectly competitive market gives the opportunity to determine the production quantity to maximize the profit using a cost function. The models were linearized and tested on a set of randomly generated instances.

Contents

1. Introduction

Nowadays, companies try to plan their business with the respect to the whole supply chain, which includes and integrates business processes from raw materials extraction through production stages and transportation activities to the end customers. The main goal of a supply chain is to integrate management activities of the supply chain members and coordinate products and information flow in order to increase its competitiveness and maximize the overall profit. At the present time, competitiveness and profitability of the whole supply chain becomes more and more important. Thereby, besides the fact that inventory management and routing are crucial parts of management activities of any organizational units in itself, their combination can help to make integrated decisions that can increase the overall benefit of the whole supply chain. Integration of routing and inventory management helps decision makers to determine the right quantity of products that has to be delivered at the right time to the right location in order to satisfy customers' needs.

Furthermore, because of globalization processes, supply chains expand and distances between actors increase, therefore, routing becomes more important and inventory becomes necessary in order to ensure robustness of a supply chain. Moreover, inventory routing problems can take place at different tiers of the supply chain, for instance transportation of raw materials between suppliers and plants or transportation of finished products between producers and retailers, this fact increases an importance of inventory routing problems even more.

Inventory routing problems are usually considered as cost minimization problems, which decrease transportation and inventory holding costs. This approach is more suitable for planning distinct processes. However, in a supply chain context, the focus of planning integrated processes should be on the profit maximization, since supply chain management strives to increase profitability of serving customers according to their needs. Profit equals revenue minus costs. Minimization of expenditures does not always lead to maximization of profit, for example, usually revenues and costs are related, therefore, minimizing costs may also minimize revenues and therefore will not maximize the profit. Vice versa, when profit is maximized the costs is not always at its minimum, for example sales that generate higher revenue costs more, however, the difference between revenue and costs is maximized. Thus, introducing profit maximization in inventory routing problems is interesting and important extension of the basic model of the inventory routing problem.

Profit maximization includes revenue that depends on prices, which in its turn depend on the type of the market. In monopolistic economy the company is a price maker, thus, it can modify its prices to maximize profits. On the other hand, in a perfect market, the firm is a price taker and cannot influence the price, however, it can choose to increase or decrease production and to not cover all demand. There are a few papers, which include profit maximization, but do not consider for example market mechanisms controlling the prices and demand (Andersson et al. 2010). Therefore, introducing profit maximization as an objective function in inventory routing problems taking into account types of markets and corresponding ways of profit maximization is an interesting topic for research.

The rest of this thesis is organized in the following way. Chapter 2 describes inventory routing problems and how profit maximization enters the picture for monopoly and perfect market situations, respectively. Chapter 3 provides a literature review, which maps out what has been done before related to this topic. In Chapter 4 models formulations with explanations are provided. Chapter 5 presents computational results and analysis. The concluding remarks of the research are provided in Chapter 6.

2. Problem description

A basic inventory routing problem (IRP) is concerned with the distribution of a single type of product from a single facility to a set of customers with given demand over a given planning horizon. The customers have inventory capacities and have to be served by capacitated homogeneous vehicles starting and ending their routes at the facility. The objective of the IRP is minimization of the sum of inventory and transportation costs without causing stockouts at any of the customers. The problem is to determine for each discrete time period the quantity to deliver to each customer and the vehicle routes. The basic model of the problem assumes that the demand is deterministic and that there is an unlimited amount of the product available at the facility (Archetti et al. 2014, Archetti et al. 2007, Campbell and Savelsbergh 2004, Coelho and Laporte 2013). An example of a basic IRP is presented in Figure 1. In this example the inventory at the supplier is limited but enough to serve all customers. The supplier has to deliver the product to 5 customers. The initial inventory, consumption (production) rate and inventory holding costs at customers and at the supplier are given.

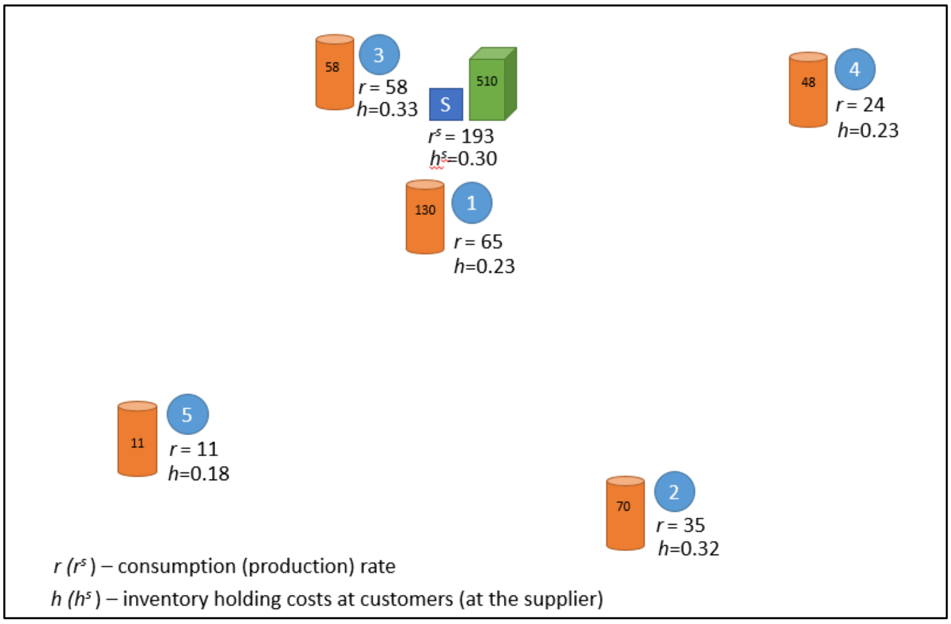

Figure 1. Example of IRP

As it was mentioned before, in a supply chain context, where managers try to increase companies' profitability, the focus of planning decisions in such an integrated problem as an IRP should be on profit maximization. An example of profit maximization case can be distribution of raw materials from a production site to consumption sites of one company or related companies, where a product will be assembled. In order to incorporate profit maximization in IRP we can simply assume that, for example, the inventory at the facility is limited and we do not have to satisfy all the demand. In this case, the objective is to maximize profit from the limited resources. However, in real life a profit maximization problem is more complicated. Ways of profit maximization depend on the type of the market, where a company operates.

In a monopoly, a company can adjust the prices to maximize profit. However, a monopolist cannot set an infinitely high price, because demand depends on prices and higher prices cause lower demand. Therefore, the profit-maximizing monopolist's problem is to find the optimal trade-off between volume and margin. The monopolist can set prices and determine the corresponding demand using the demand function (Figure 2). If we take into account a production stage in addition to the inventory and routing decisions, profit equals revenue minus inventory, transportation and production costs. Revenue in its turn can be found as price multiplied by production quantity. However, in a monopoly, price and quantity are decision variables. Therefore, the objective function becomes non-linear.

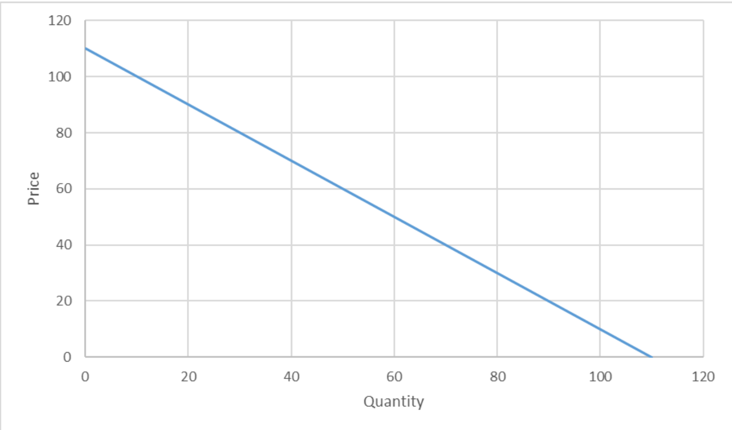

Figure 2. Demand function

In a perfectly competitive market, a company is a price-taker and cannot influence the price. However, unit costs vary with production volume. Thus, a firm can determine production quantity to maximize its profit using a cost function (Figure 3). In a perfect market, profit also equals revenue minus inventory, transportation and production costs. However, the price is fixed in this case. The production costs is defined as a product of unit costs and the production quantity. In a perfect market situation, unit costs and production quantity are variables that makes the objective function non-linear as well.

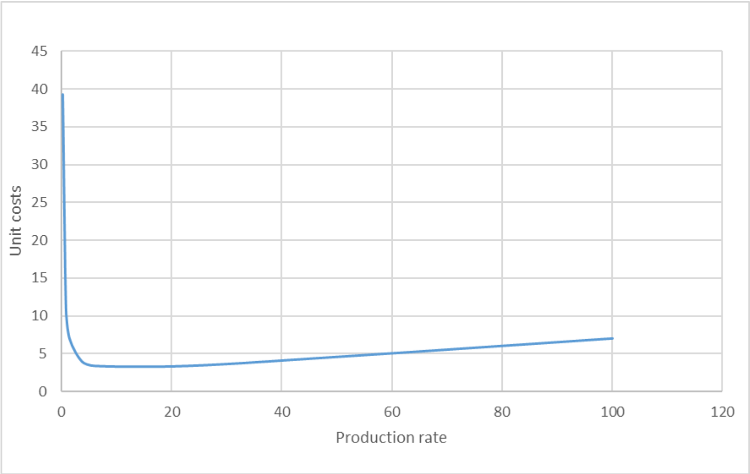

Figure 3. Cost function

Non-linear problems are more difficult to solve. Thus, the problem is to incorporate and linearize profit maximization in IRP with respect to the type of the decision-maker's market. That can help companies to make better decisions taking into account several planning aspects at the same time.

3. Literature review

Operations research literature, which belongs to such academic disciplines as economic theory, management science and business administration, will be relevant for this research problem. First, it is necessary to look at general inventory routing problems, then, focus on IRPs with profit maximization. Finally, production routing problems should be considered as we take into account a production stage with respect to an opportunity to adjust the production amount.

3.1. Inventory routing problems

In the beginning of the literature search, it is useful to look for some existing literature reviews related to the research problem. There are several literature surveys of inventory routing problems (Andersson et al. 2010, Coelho, Cordeau, and Laporte 2014, Moin and Salhi 2007).

Moin and Salhi (2007) present an overview of Supply Chain Management focusing on the inventory routing area. It highlights the helpfulness and restrictions of the models in practice. Moin and Salhi (2007) have classified the papers based on the planning horizon considered in the models namely single period, multiperiod and infinite horizon models with deterministic and stochastic demand patterns. Future research directions are also presented.

Andersson et al. (2010) provide an overview of combined inventory management and routing problems, describes industrial aspects and gives a classification and comprehensive literature review of the current state of the research. Based on the status and trends within the field, future research is suggested with regard to both further development of the research area and industrial needs.

Coelho, Cordeau, and Laporte (2014) provide a comprehensive review of the literature related to inventory routing problems. Coelho, Cordeau, and Laporte (2014) categorize inventory routing problems with respect to their structural variants and the availability of information on customer demand. The structural variants include such criteria as time horizon (finite or infinite), structure (one-to-one, one-to-many or many-to-many), routing (direct, multiple or continuous), inventory policy (maximum level or order-up-to-level), inventory decisions (lost sales, backorders nonnegative), fleet composition (homogeneous or heterogeneous), and fleet size (single, multiple or unrestricted).

The inventory routing problems with cost minimization can be taken as a basis for incorporating a profit maximization objective function. Archetti et al. (2007) present a vendor-managed inventory routing problem with different types of replenishment policy. The problem is to determine for each time period the quantities of a product to ship to each customer with defined maximum inventory levels and the routes of the vehicles with given capacities. A branch-and-cut algorithm is implemented to solve the model with a minimization objective function. The model is tested on a set of randomly generated instances.

Archetti, Desaulniers, and Speranza (2016) consider a non-linear objective function in inventory routing problems with a finite time horizon trying to avoid a drawback of zero inventory at customers at the end of the time horizon. The objective function is the minimization of the logistic ratio, which is the ratio of the total transportation costs to the total delivered quantity. The results are compared to those of a classical IRP.

Archetti et al. (2014) provide and analyze different mathematical programming formulations of a multi-vehicle IRP such as vehicle-indexed formulations and flow formulations. The objective function is the minimization of transportation and inventory holding costs. The formulations are tested on a set of instances.

3.2. Inventory routing problems with profit maximization

The next step is to search the literature that is the closest to the research problem. In this case the most relevant topic is inventory routing problems with profit maximization. There are several articles related to the presented topic (Andersson, Christiansen, and Fagerholt 2010, Chien, Balakrishnan, and Wong 1989, Fodstad et al. 2010, Grønhaug et al. 2010, Papageorgiou et al. 2014, Bell et al. 1983). Most of these articles consider inventory routing problems in liquefied natural gas (LNG) supply chain (Andersson, Christiansen, and Fagerholt 2010, Fodstad et al. 2010, Grønhaug et al. 2010).

Chien, Balakrishnan, and Wong (1989) provide the problem of distributing a limited amount of inventory among customers using a fleet of vehicles to maximize profit. The problem consists of a central depot with fixed supply capacities and many customers with deterministic demand. The entire demand need not be satisfied but there is a penalty cost imposed per unit of unsatisfied demand. The objective is to maximize profit that consists of total revenue less the penalty cost and routing costs. They formulate the integrated inventory and routing problem as a mixed integer program and develop a Lagrangian-based procedure to generate both good upper bounds and heuristic solutions.

Andersson, Christiansen, and Fagerholt (2010) introduce the LNG supply chain and two planning problems related to the transportation planning and inventory management within the chain. One of the planning problems is to sequence and schedule voyages and to assign them to ships. The objective function is to minimize the cost of operating the voyages and the cost of over- and under-deliveries. The second problem is to design routes and schedules including determining sales volumes that maximize the company revenue from the sales minus the operational costs. In this model three types of contracts are considered: the first one is a fixed contract where the volume cannot be violated, the second one includes lower and upper limits for delivery quantities, the third one is a short-term contract which should be satisfied only if profitable. Both problems are formulated as mixed integer programs, and possible solution methods are briefly discussed.

Fodstad et al. (2010) present an optimization model that provides decision support for the liquefied natural gas supply chain by coordinating vessel routing, inventory management (upstream, onboard and downstream), trading and contract obligations. The model maximizes profit by utilizing different trading contracts. Contracts can have upper and lower quantity limits within any user-defined time window.

Grønhaug et al. (2010) consider a maritime inventory routing problem in the liquefied natural gas (LNG) business, called the LNG inventory routing problem (LNG-IRP). Here, a producer is responsible for the routing of the fleet of ships, and the inventories both at the liquefaction plants and the regasification terminals. Authors describe features of the LNG-IRP compared to other maritime inventory routing problems. The problem is solved by a branch-and-price method and the column generation approach. The presented model maximizes total profit, which consists of sales revenues minus the production and transportation costs. The sales and production quantities are bounded by the interval, and unit sales revenues and production costs are given. The proposed method is tested on real-world instances.

Papageorgiou et al. (2014) present a detailed description of deterministic single product maritime inventory routing problems (MIRPs), which are called deep-sea MIRPs with inventory tracking at every port. The paper introduces a model for it as a mixed-integer linear program. The objective function is to maximize revenue minus travel costs, while production/sales quantities are limited within the predefined intervals. Papageorgiou et al. (2014) present a library, called MIRPLib, of publicly available test problem instances for MIRPs with inventory tracking at every port.

Bell et al. (1983) consider inventory management of industrial gases at customer locations combined with vehicle scheduling and dispatching. The paper introduces the mathematical model that maximizes revenue minus transport costs, which includes mileage of route, drivers pay regulations, fuel costs, and vehicle depreciation per mile. The model with profit maximization to produce daily delivery schedules is solved using a sophisticated Lagrangian relaxation algorithm.

To sum up how previous literature has considered profit maximization in inventory routing the main characteristics must be defined. All of the articles that have been found on this topic provide a mathematical model of an IRP with a maximization objective. All of the models are multiperiod, contain fixed lower and upper bounds for the customers' demand with associated unit sales revenue for each period and allow the sales/delivery quantities to be decided (Andersson, Christiansen, and Fagerholt 2010, Grønhaug et al. 2010, Chien, Balakrishnan, and Wong 1989, Fodstad et al. 2010, Papageorgiou et al. 2014, Bell et al. 1983). In some articles the lower and upper limits for the inventory level are given (Andersson, Christiansen, and Fagerholt 2010, Fodstad et al. 2010, Grønhaug et al. 2010, Papageorgiou et al. 2014), Chien, Balakrishnan, and Wong (1989) assume that only the supplier has inventory capacities. In the article written by Bell et al. (1983) the model does not contain inventory balance constraints, however parameters of maximum and minimum amount of a product that can be delivered take into account inventory levels that are calculated by demand and inventory calculator outside the model. None of the described models contains inventory holding costs in the objective function. Most of the articles consider maritime inventory routing problems and assume that the shipper owns both the production and consumption sites and inventory holding costs are the same (Andersson, Christiansen, and Fagerholt 2010, Fodstad et al. 2010, Grønhaug et al. 2010).

Most of the articles consider homogeneous fleet of vehicles; however, some of them include a heterogeneous fleet (Fodstad et al. 2010, Grønhaug et al. 2010). There are some other features. Several articles contain a production stage; the models allow determining the production amount within a predefined interval with fixed production unit costs (Grønhaug et al. 2010, Papageorgiou et al. 2014). Fodstad et al. (2010) consider time windows and different types of contracts with different fixed purchase prices. Some articles take into account a decision variable that represents the amount of product purchased from the spot market (Fodstad et al. 2010, Grønhaug et al. 2010, Papageorgiou et al. 2014). Chien, Balakrishnan, and Wong (1989) introduce penalty for the demand that is not satisfied. However, none of the articles consider important aspects of profit maximization such as possibility of the prices adjustment and the unit production costs variation with production volume. The literuture overview is presented in Table 1.

Table 1 - Literature overview of inventory routing problems with profit maximization

		Andersson, Christiansen, and Fagerholt 2010	Bell et al. 1983	Chien, Balakrishnan, and Wong 1989	Fodstad et al. 2010	Grønhaug et al. 2010	Papageorgiou et al. 2014
math model with max profit		+	+	+	+	+	+
multiperiod		+	+	+	+	+	+
fixed lower and upper bounds for the customers' demand		+	+	+	+	+	+
unit sales revenue for each period		+	+	+	+	+	+
allow the sales/delivery quantities to be decided		+	+	+	+	+	+
the given lower and upper limits for the inventory level		+	Calculated outside the model	Only at the supplier	+	+	+
vehicle fleet	homogeneous	+	+	+			+
	heterogeneous				+	+	
contain a production stage, determining the production amount within a predefined interval						+ (with fixed production unit costs)	+
time windows					+		
different types of contracts with different fixed purchase prices					+		
the spot market					+	+	+
penalty for the demand that is not satisfied				+			

3.3. Production routing problems

It is useful to look for the existing survey of production routing problems to understand the main idea and different formulation schemes of the PRP. There is a literature review related to this topic (Adulyasak, Cordeau, and Jans 2015). The paper states that the

PRP contains both lot-sizing and vehicle routing solutions and helps to jointly optimize production, inventory, distribution and routing decisions. Therefore, the PRP can be considered as a generalization of the IRP. The article provides a comprehensive review of different solution techniques that are used to solve the PRP. According to the article, even if production stage takes place, the objective function is cost minimization. The costs in this case include the total production, setup, inventory and routing costs.

During the literature research it was noticed that there are a lot of literature related to inventory routing problems, most of the articles include models with cost minimization as an objective function and just a few of them consider profit maximization. Even when taking into account production decisions in addition to inventory management and routing, one still does not consider profit maximization. As it was mentioned before, an inventory routing problem with profit maximization is an important problem in itself, in addition, lack of the literature about this topic means that this field needs further research and extension of existing models by taking into account different planning aspects.

4. Models formulation

In this chapter mathematical models of IRP will be presented. First, IRP with cost minimization will be formulated. Second, the model will be modified to the one with profit maximization objective function. Next, the models with profit maximization for two types of market (monopoly and perfect competition) will be developed and ways of their linearization will be provided. Finally, route generation algorithm will be described.

4.1. Model 1. Inventory routing problems with cost minimization

In this work the notation presented by Archetti et al. (2007) will be used as a basis and modified in order to fit the problem.

Let us consider an inventory routing problem for a logistic network where a single type of product is shipped from one supplier 0 to a set of customers N over a time horizon T. The supplier uses a maximum level inventory policy where the shipping quantity must be not greater than the inventory capacity of customers. The supplier has a maximum inventory level U^s, inventory holding costs h^s, an initial inventory level B^0 and a production rate at each time period r_t^s. Unit production costs are defined by a unit costs function $f(r_t^s)$. At each time period $t \in T = \{1, \dots, t\}$ customers consume an amount of product r_i where $i \in N$. Each customer defines a maximum inventory level U_i and has an initial inventory level I_i^0 and inventory holding costs h_i. An inventory level at the end of time period t at the supplier and customers is denoted as variables B_t and I_{it} respectively. The product has to be shipped by a homogeneous fleet of vehicles of capacity Q. Parameter n defines a number of available vehicles, which should perform a delivery using a set of routes $K = \{1, 2, \dots, k\}$ with costs c_k. A binary parameter a_{ik} equals 1 if customer i is served on route k, 0 otherwise. Each vehicle can perform no more than one route per day. Denoting by Y_{kt} we introduce a binary variable equal to 1 if route k is used at time t and 0 otherwise. Supposing that a variable X_{ikt} identifies a quantity of product shipped to customer i at time period t using route k and deliveries take place before the consumption we can formulate a mathematical model so that transportation and total inventory holding costs are minimized.

$$min \sum_{t\in T} \sum_{k\in K} c_k\, Y_{kt} + \sum_{i\in N} \sum_{t\in T} h_i\, I_{it} + \sum_{t\in T} h^s\, B_t\, r_t^s + \sum_{t\in T} f(r_t^s) r_t^s \qquad (1.1)$$

s.t.

$$\sum_{i\in N} X_{ikt} \le Q\, Y_{kt} \qquad t \in T, k \in K \qquad (1.2)$$

$$X_{ikt} \le Q\, a_{ik} Y_{kt} \qquad t \in T, i \in N, k \in K \qquad (1.3)$$

$$\sum_{k\in K} Y_{kt} \le n \qquad t \in T \qquad (1.4)$$

$$I_{i0} = I_i^0 \qquad i \in N \qquad (1.5)$$

$$I_{it} = I_{i,t-1} + \sum_{k\in K} X_{ikt} - r_i \qquad i \in N, t \in T \qquad (1.6)$$

$$\sum_{k\in K} X_{ikt} \le U_i - I_{i,t-1} \qquad i \in N, t \in T \qquad (1.7)$$

$$B_t = B_{t-1} + r_t^s - \sum_{i\in N} \sum_{k\in K} X_{i,k,t} \quad t \in T \qquad (1.8)$$

$$B_{t-1} + r_t^s \le U^s \qquad t \in T \qquad (1.9)$$

$$B_0 = B^0 \qquad (1.10)$$

$$I_{it} \ge 0 \qquad i \in N, t \in T \qquad (1.11)$$

$$B_t \ge 0 \qquad t \in T \qquad (1.12)$$

$$X_{ikt} \ge 0 \qquad i \in N, k \in K, t \in T \qquad (1.13)$$

$$Y_{kt} \in \{0,1\} \qquad k \in K, t \in T \qquad (1.14)$$

The objective function (1.1) expresses a minimization of the total costs, which include transportation costs, total inventory holding costs at customers, total inventory holding costs at the supplier and total production costs. The total production costs are fixed in this case, so it does not influence the objective function. Constraints (1.2) ensure that the quantity delivered by a vehicle is not greater than its capacity. Constraints (1.3) guarantee that a delivery at each time period takes place only if a customer is visited with a route and this route is used at this time period. The constraints (1.4) limit the number of routes per time period by the number of available vehicles. An initial inventory level at customers is determined by constraints (1.5). Constraints (1.6) define an inventory level at customers at each time period. Inventory level at customers at time period *t* equals the inventory level at the previous period plus the quantity of the product delivered at this time period minus consumption rate of the customer. Constraints (1.7) ensure that an inventory level at customers will not exceed its maximum level. Constraints (1.8) determine an inventory level at the supplier. The inventory level at the supplier at the current period of time equals the inventory level at the previous time period plus production quantity at this time period minus the total volume delivered to all customers at this time period. Constraint (1.9) limits an inventory level at the supplier by its maximum. Constraint (1.10) defines an initial inventory level at the supplier. Constraints (1.11) – (1.14) are negativity and integrality constraints.

4.2. Model 2. Inventory routing problems with profit maximization

Let us consider a profit maximization case of the previous problem. In this case the product distribution from a production site to consumption sites of one company or different related companies will be considered. However, in order to keep conventional terminology the terms "supplier" and "customer" will be used.

In a profit maximization case the supplier can get a sales revenue p_i per unit of product shipped to customers, which is a unit price. It is not necessary to satisfy all the demand of customers, so the demand can be partially lost. However, there is a penalty b_i for each unit of the unsatisfied demand that helps to take into account customers' needs. The problem is to maximize the overall profit. Since the consumption amount can be less than the demand we introduce a variable C_{it} that identifies the amount of product consumed by customer i at period of time t. The mathematical model is presented below.

$$\max \sum_{i\in N}\sum_{k\in K}\sum_{t\in T} p_i X_{ikt} - \sum_{t\in T}\sum_{k\in K} c_k\, Y_{kt} - \sum_{i\in N}\sum_{t\in T} h_i\, I_{it} - \sum_{t\in T} h^s\, B_t - \sum_{i\in N}\sum_{t\in T} b_i(r_i - C_{it}) - \sum_{t\in T} f(r_t^s) r_t^s \quad (2.1)$$

s.t.

$$\sum_{i\in N} X_{ikt} \le Q\, Y_{kt} \qquad t\in T, k\in K \quad (2.2)$$

$$X_{ikt} \le Q\, a_{ik} Y_{kt} \qquad t\in T, i\in N, k\in K \quad (2.3)$$

$$\sum_{k\in K} Y_{kt} \le n \qquad t\in T \quad (2.4)$$

$$I_{i0} = I_i^0 \qquad i\in N \quad (2.5)$$

$$I_{it} = I_{i,t-1} + \sum_{k\in K} X_{ikt} - C_{it} \qquad i\in N, t\in T \quad (2.6)$$

$$C_{it} \leq r_i \qquad i \in N, t \in T \tag{2.7}$$

$$\sum_{k \in K} X_{ikt} \leq U_i - I_{i,t-1} \qquad i \in N, t \in T \tag{2.8}$$

$$B_t = B_{t-1} + r_t^s - \sum_{i \in N} \sum_{k \in K} X_{i,k,t} \quad t \in T \tag{2.9}$$

$$B_{t-1} + r_t^s \leq U^s \qquad t \in T \tag{2.10}$$

$$B_0 = B^0 \tag{2.11}$$

$$I_{it} \geq 0 \qquad i \in N, t \in T \tag{2.12}$$

$$B_t \geq 0 \qquad t \in T \tag{2.13}$$

$$X_{ikt} \geq 0 \qquad i \in N, k \in K, t \in T \tag{2.14}$$

$$Y_{kt} \in \{0,1\} \qquad k \in K, t \in T \tag{2.15}$$

$$C_{it} \geq 0 \qquad i \in N, t \in T \tag{2.16}$$

The objective function (2.1) maximizes the total profit equal to the total revenue minus transportation, total inventory holding costs at customers and at the supplier, penalty for the unsatisfied demand and total production costs. Constraints (2.6) are an inventory balance constraints. The inventory level is defined as the inventory level at the previous period plus shipped amount of product minus consumed amount of product. The consumed amount of product must be not greater than the demand of a customer. It is stated by constraints (2.7). The rest of the constraints are the same as in model 1.

4.3. Model 3. Inventory routing problems with profit maximization for monopoly

In a monopoly, a company can adjust the prices to maximize profit. However, a monopolist cannot set an infinitely high price, because demand depends on prices and higher prices cause lower demand. In this case the unit revenue becomes a variable P_i . The dependency of the demand on the unit price is described by a function $r_i = f(P_i)$. All the constraints of the model 2 remain the same except the constraints (3.7), which now have a function as a right hand side.

$$\max \sum_{i\in N}\sum_{k\in K}\sum_{t\in T} P_i X_{ikt} - \sum_{t\in T}\sum_{k\in K} c_k\, Y_{kt} - \sum_{i\in N}\sum_{t\in T} h_i\, I_{it} - \sum_{t\in T} h^s\, B_t$$
$$- \sum_{i\in N}\sum_{t\in T} b_i\big(f(P_i) - C_{it}\big) - \sum_{t\in T} f(r_t^s) r_t^s \qquad (3.1)$$

s.t.

$$\sum_{i\in J} X_{ikt} \le Q\, Y_{kt} \qquad t \in T, k \in K \qquad (3.2)$$

$$X_{ikt} \le Q\, a_{ik} Y_{kt} \qquad t \in T, i \in N, k \in K \qquad (3.3)$$

$$\sum_{k\in K} Y_{kt} \le n \qquad t \in T \qquad (3.4)$$

$$I_{i0} = I_i^0 \qquad i \in N \qquad (3.5)$$

$$I_{it} = I_{i,t-1} + \sum_{k\in K} X_{ikt} - C_{it} \qquad i \in N, t \in T \qquad (3.6)$$

$$C_{it} \le f(P_i) \qquad i \in N, t \in T \qquad (3.7)$$

$$\sum_{k\in K} X_{ikt} \le U_i - I_{i,t-1} \qquad i \in N, t \in T \qquad (3.8)$$

$$B_t = B_{t-1} + r_t^s - \sum_{i\in N}\sum_{k\in K} X_{i,k,t} \quad t \in T \qquad (3.9)$$

$$B_{t-1} + r_t^s \le U^s \qquad t \in T \qquad (3.10)$$

$$B_0 = B^0 \qquad (3.11)$$

$$I_{it} \ge 0 \qquad i \in N, t \in T \qquad (3.12)$$

$$B_t \ge 0 \qquad t \in T \qquad (3.13)$$

$$X_{ikt} \ge 0 \qquad i \in N, k \in K, t \in T \qquad (3.14)$$

$$Y_{kt} \in \{0,1\} \qquad k \in K, t \in T \qquad (3.15)$$

$$C_{it} \ge 0 \qquad i \in N, t \in T \qquad (3.16)$$

$$P_i \ge 0 \qquad i \in N \qquad (3.17)$$

In this model the demand function that has a form $f(P_i) = bP_i + d$ will be considered (Besanko and Braeutigam 2010). The demand function is linear and does not create any difficulties in constraints (3.7). However, the objective function (3.1) becomes non-linear and non-separable as it is a product of two variables: price and shipped quantity.

The importance of separable functions is that they can be approximated to by piecewise linear functions. Then it is possible to use separable programming.

It is often possible to transform the model with non-separable functions into one with only separable functions. In our case we have to convert the product of two variables into a separable form. However, before the conversion in order to avoid indices for routes and time periods we introduce a new variable $Z_i = \sum_{k\in K, t\in T} X_{ikt}$ for each $i \in N$, which is the sum of the shipped amount of product over routes and time periods. Now the term in the objective function that we need to convert is $\sum_{i\in J} P_i Z_i$. In order to convert the product of two variables we need to perform the following transformations (Williams 2013). First, we introduce two new variables W_{1i} and W_{2i} into the model. Second, we relate the new variables W_{1i} and W_{2i} to P_i and Z_i by the following relations:

$$W_{1i} = \frac{1}{2}(P_i + Z_i)$$

$$W_{2i} = \frac{1}{2}(P_i - Z_i)$$

If $l^P \leq P_i \leq u^p$ and $l^Z \leq Z_i \leq u_i^Z$, then the bounds on W_{1i} and W_{2i} are:

$$\frac{1}{2}(l^P + l^Z) \leq W_{1i} \leq \frac{1}{2}(u^P + u_i^Z)$$

$$\frac{1}{2}(l^P - u_i^Z) \leq W_{2i} \leq \frac{1}{2}(u^P - l^Z)$$

Then we replace the term $\sum_{i\in N} P_i Z_i$ in the objective function by $\sum_{i\in N}(W_{1i}^2 - W_{2i}^2)$, which is a separable function as it contains non-linear functions of a single variable. These non-linear terms can be eliminated by piecewise linear approximations.

This approximation can be performed in several ways. In our model a method known as the λ-formulation will be used.

Let $w1_{is}$ where $i \in N, s \in \{1,..,S^W\}$ denote breakpoints for the function $g(W_{1i}) = W_{1i}^2$ with the number of points equal to S^W and $w2_{is}$ where $i \in N, s \in \{1,..,S^W\}$ denote breakpoints for the function $g(W_{2i}) = W_{2i}^2$ with the number of points equal to S^W. Then, let $g(w1_{is})$ and $g(w2_{ij})$ denote the corresponding function values. By these breakpoints the curves are divided into pieces that are approximated by straight lines. Any point between two breakpoints is a weighted sum of these two points. A schematic graphical representation of the approximation of the function $g(W_{1i}) = W_{1i}^2$ for $i \in N$ is demonstrated in Figure 3.

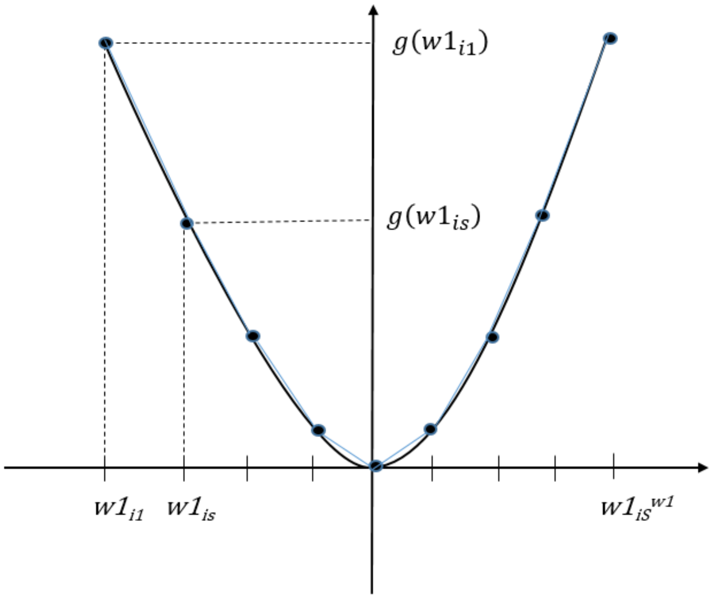

Figure 3. Piecewise linear approximation

Let λ_{is}^{w1} where $i \in N, s \in \{1,..,S^W\}$ and λ_{is}^{w2} where $i \in$ N, $s \in \{1,..,S^W\}$ denote nonnegative weights for the function $g(W_{1i})$ and $g(W_{2i})$ correspondingly. Then, the piecewise linear approximation can be written as following:

$$max \sum_{i \in N} \sum_{s \in \{1,..,S^W\}} \lambda_{is}^{w1} \, g(w1_{is}) \; - \sum_{i \in N} \sum_{s \in \{1,..,S^W\}} \lambda_{is}^{w2} \, g(w2_{is})$$

$$\sum_{s \in \{1,..,S^W\}} \lambda_{is}^{w1} \, w1_{is} = W_{1i} \quad i \in N$$

$$\sum_{s \in \{1,..,S^W\}} \lambda_{is}^{w1} = 1 \quad i \in N$$

$$\sum_{s \in \{1,..,S^W\}} \lambda_{is}^{w2} \, w2_{is} = W_{2i} \quad i \in N$$

$$\sum_{s \in \{1,..,S^W\}} \lambda_{is}^{w2} = 1 \quad i \in N$$

An additional requirement is that at most two adjacent λ_{is}^{w1} can be greater than zero. This class of constraint is known as a special ordered set of type 2 (SOS2). The requirement guarantees that corresponding values of W_{1i} and $g(W_{1i})$ always lie on one of the straight line segments between breakpoints. This added stipulation can be modeled using additional binary variables. However, integer (binary) programming is generally more costly in computer time. Therefore, it should be used only if it is necessary. The added adjacency requirements are redundant in case of minimizing convex functions or maximizing the negation of a convex function.

In our case, the term $g(W_{2i}) = \sum_{i \in N}(-W_{2i}^2)$ does not need additional constraints, because we maximize the negation of a convex function. However, the term $g(W_{1i}) = \sum_{i \in N} W_{1i}^2$ produces some difficulties as we maximize a convex function. Thus, we need to add binary variables S_{is} where $i \in \mathrm{N}, s \in \{1, .., S^W - 1\}$ that represent the intervals between two adjacent breakpoints and equal 1 if the interval is chosen and 0 otherwise. Only one interval can be chosen, that is guaranteed by the following constraint:

$$\sum_{s \in \{1,..,S^W - 1\}} S_{is} = 1 \qquad i \in N$$

The next constraints connect intervals and corresponding breakpoints.

$$\lambda_{i1}^{w1} \leq S_{i1} \qquad i \in N$$

$$\lambda_{is}^{w1} \leq S_{i,s-1} + S_{is} \qquad i \in N, s \in \{2, .., S^W - 1\}$$

$$\lambda_{iS^w}^{w1} \leq S_{i,S^w - 1} \qquad i \in N$$

Instead of adding binary variables a solver can be provided with the information that the set of variables is a special ordered set of type 2. In this case the solver will be set up to use SOS2 branching.

In the way described above the model with the product of two variables in the objective function can be linearized. However, the cost of this linearization is an approximated value of the objective function. A degree of the approximation depends on the number of the breakpoints: the more breakpoints we have the closer approximation we get. However, if we increase the number of breakpoints, it will increase the time that is needed to solve the model.

4.4. Model 4. Inventory routing problems with profit maximization for perfect competition

In a perfectly competitive market, a company is a price-taker and cannot influence the price. However, unit costs vary with production volume and it is described by the function $f(R_t^s)$. We introduce a variable R_t^s, which is production quantity.

$$\max \sum_{i\in N}\sum_{k\in K}\sum_{t\in T} p_i X_{ikt} - \left(\sum_{t\in T}\sum_{k\in K} c_k\, Y_{kt} + \sum_{i\in N}\sum_{t\in T} h_i\, I_{it} + \sum_{t\in T} h^s\, B_t\right) - \sum_{i\in N}\sum_{t\in T} b_i (r_i - C_{it}) - \sum_{t\in T} f(R_t^S) R_t^S \tag{4.1}$$

s.t.

$$\sum_{i\in N} X_{ikt} \le Q\, Y_{kt} \qquad t\in T, k\in K \tag{4.2}$$

$$X_{ikt} \le Q\, a_{ik} Y_{kt} \qquad t\in T, i\in N, k\in K \tag{4.3}$$

$$\sum_{k\in K} Y_{kt} \le n \qquad t\in T \tag{4.4}$$

$$I_{i0} = I_i^0 \qquad i\in N \tag{4.5}$$

$$I_{it} = I_{i,t-1} + \sum_{k\in K} X_{ikt} - C_{it} \qquad i\in N, t\in T \tag{4.6}$$

$$C_{it} \le r_i \qquad i\in N, t\in T \tag{4.7}$$

$$\sum_{k\in K} X_{ikt} \le U_i - I_{i,t-1} \qquad i\in N, t\in T \tag{4.8}$$

$$B_t = B_{t-1} + R_t^S - \sum_{i\in N}\sum_{k\in K} X_{i,k,t} \quad t\in T \tag{4.9}$$

$$B_{t-1} + R_t^S \le U^s \qquad t\in T \tag{4.10}$$

$$B_0 = B^0 \tag{4.11}$$

$$R_t^S \ge 0 \qquad t\in T \tag{4.12}$$

$$I_{it} \ge 0 \qquad i\in N, t\in T \tag{4.13}$$

$$B_t \ge 0 \qquad t\in T \tag{4.14}$$

$$X_{ikt} \ge 0 \qquad i\in N, k\in K, t\in T \tag{4.15}$$

$$Y_{kt} \in \{0,1\} \qquad k\in K, t\in T \tag{4.16}$$

$$C_{it} \ge 0 \qquad i\in N, t\in T \tag{4.17}$$

The term $\sum_{t\in T} f(R_t^S) R_t^s$ in the objective function is the total production costs, which is the total average costs multiplied by the production quantity. A function of the total average costs has the form $f(R_t^s) = eR_t^s + d + \frac{m}{R_t^s}$ (Besanko and Braeutigam 2010). If we multiply the function of the total average costs by the production quantity the function of the total production costs will have the following form:

$$f(R_t^S) R_t^s = e{R_t^s}^2 + dR_t^s + m$$

Then, the term in the objective function $\sum_{t\in T}(eR_t^{s2} + dR_t^s + m)$ is a separable non-linear function. In order to linearize the model we have to eliminate non-linear function of a single variable. It can be done using λ-formulation method of piecewise linear approximation that was described above.

Let w_{ts} where $t \in T, s \in \{1,..,S^W\}$ denote breakpoints for the function $g(R_t^s) = \sum_{t\in T} eR_t^{s2}$ with the number of points equal to $S^W$and $g(w_{ts})$ denote the corresponding function values. Let λ_{ts} where $t \in T, s \in \{1,..,S^W\}$ denote nonnegative weights for the function $g(R_t^s)$. As in this case we minimize a convex function, the adjacency requirements are redundant. Then, the piecewise linear approximation can be written as following:

$$g(R_t^s) = \sum_{t\in T} \sum_{s\in\{1,..,S^W\}} \lambda_{ts}\, g(w_{ts})$$

$$\sum_{s\in\{1,..,S^W\}} \lambda_{ts}\, w_{ts} = R_t^s \quad t \in T$$

$$\sum_{s\in\{1,..,S^W\}} \lambda_{ts} = 1 \quad t \in T$$

4.5. Route generation algorithm

The two-phase method will be used for solving IRP problems. On the first phase the generation of routes will be performed as a sub problem. On the second phase the described IRP models will be used as master models.

In order to generate the set of possible routes for the models a route generation algorithm will be used. First, using coordinates as input data we calculate the distances between all nodes (including customers and the supplier). Second, we define all possible combinations (subsets) of customers up to a certain maximum number of customers per route. Then, for each subset of customers we solve a travelling salesman problem (TSP) by finding the permutation of customers with the shortest distance of the route. The result of the route generation that we can use in the models is a set of shortest routes with the costs of the routes and a binary parameter, which equals 1 if route k includes customer i, 0 otherwise.

5. Computational experiments

In this chapter generation of instances will be described and computational results and analysis will be presented.

All computational tests were run on a personal computer with 2.50 GHz Intel Core i5-6500T CPU and 16 GB of RAM under Microsoft Windows 10 Enterprise 64-bit version. The models were tested with AMPL/CPLEX 12.7.00.

5.1. Generation of instances

In order to test the models, understand the technical and economical behavior of the models and the maximum size of the problems that can be solved using the models within reasonable time test instances were generated.

The test instances were generated on the basis of the test instances presented by Archetti et al. (2007) which were modified in order to fit the problem.

The values of parameters were assumed as following. The time horizon T consists of 3 and 6 time periods. The considered number of customers N is 5, 10 and 15. The product quantity r_i consumed by customer i at time t is randomly generated as an integer number in the interval [10, 100]. The production rate r^s is the sum of consumption rates of customers $(\sum_{i \in J} r_i)$. The maximum inventory level U_i at customers equals $r_i g$, where $g \in \{2,3\}$ and indicates the number of time periods needed to consume the amount U_i . The maximum inventory level U^s at the supplier equals the sum of maximum inventory levels at customers multiplied by 2 $(2 \sum_{i \in J} U_i)$. The starting inventory level at customers I_i^0 is the maximum inventory level at customers minus consumption rate $(U_i - r_i)$. The starting inventory level at the supplier B^0 is the sum of maximum inventory levels at customers $(\sum_{i \in J} U_i)$. The inventory holding costs h^s at the supplier are 0.3 and the inventory holding costs h_i at customers are randomly generated in the interval [0.1, 0.5]. The vehicle capacity Q is $\frac{1.5}{n} \sum_{i \in J} r_i$ where n is a number of available vehicles. The coordinates (Xi,Yi) of customers and the supplier are randomly generated in the interval [0,500] and transportation costs are calculated as $\sqrt{(X_0 - X_i)^2} + \sqrt{(Y_0 - Y_i)^2}$. The maximum number of customers on each route is 2 and 3. The number of vehicles is 3. The demand function is $f(P_i) = -2.5P_i + 113$, where P_i is a unit price. The unit price limit for the monopoly: 41, with the corresponding demand 10.5. The penalty for unsatisfied demand is $0.2p_i$ for model 1, 2 and 4. For a monopoly where the price is variable the value of penalty is assumed equal to the absolute value of penalty in other models. This assumption is made to simplify the model and to avoid additional non-linearity in the objective function. The average costs function is $f(R_t^s) = 0.0005R_t^s + 2 + \frac{3}{R_t^s}$, where R_t^s is a production rate. The number of breakpoints for piecewise linear approximation is 5, 10 and 15.

5.2. Computational results

In order to test the model a number of experiments were conducted. The computational tests can be divided into two groups according to the considered criteria: technical and economical.

5.2.1. Technical tests

In order to study the models from technical point of view we solve instances for 3 and 6 periods and 5, 10 and 15 customers with the number of breakpoints (for model 3 and 4) equal to 5 and maximum 2 customers per route. The computational time of the instances is presented in table 2. The AMPL codes for model 3 and 4 are presented in appendix 1 and 2 respectively.

Table 2 - Computational time in seconds

n	Model 1	Model 2	Model 3	Model 4
T3				
5	0.13	0.24	0.39	0.22
10	0.25	0.55	11.01	0.58
15	0.41	42.91	158.03	24.64
T6				
5	0.41	0.23	22.28	0.22
10	0.44	0.69	446.58	0.67
15	0.27	31.17	2863.16 (with 10% gap)	47.25

The computational time increases with the increase of the problems size. The instances up to 6 periods and 15 customers can be solved within 1 minute using model 1, 2 and 4.

The running time of model 3 increases significantly because of the additional integer variables. The instance for 6 periods and 10 customers is solved optimally within 8 minutes, however, if we increase the number of customers up to 15 it takes about 48 minutes to solve the problem with 10% gap.

The computational time and the outcome of the model are also influenced by the number of breakpoints. Let us consider the small instance for 3 periods, 5 customers, and maximum 2 customers per route with 5, 10 and 15 breakpoints (S^W) to understand the influence of the number of breakpoints on model 3 for a monopoly. The computational results are presented in table 3.

Table 3 - Number of breakpoints

Instance	Criterion	S^W=5	S^W=10	S^W=15
T3n5	approximated profit	12476.30	11737.40	11567.30
	real profit	10728.22	11343.04	11342.90
	deviation from real profit	16 %	3 %	2 %
	approximated revenue	17038.50	17073.50	16499.50
	real revenue	15290.40	16679.20	16275.10
	deviation from real revenue	11 %	2 %	1 %
	costs	4562.18	5336.16	4932.20
	transportation costs	2504.49	3423.34	2897.34
	inventory holding costs at customers	206.27	183.88	195.37
	inventory holding costs at the supplier	608.15	506.07	526.86
	penalty	20.40	0.00	89.76
	production costs	1222.87	1222.87	1222.87
	produced amount	579	579	579
	shipped amount of product	473	603	574
	consumed amount of product	421	617	576
	time (seconds)	0.39	0.80	1.58

The increase of the number of breakpoints from 5 to 15 the deviation of approximated value of revenue and profit from the real ones reduces from 11% to 1% and from 16% to 2% correspondingly. We get better approximation, however we can see that the computational time increases.

As it was mentioned before, another way of the piecewise linear approximation formulation is to provide the solver with the information that the set of variables is SOS2. To compare two alternative formulations of model 3 the instance for 3 and 6 periods, 10 customers, 5 breakpoints and maximum 2 customers per route will be solved. The results are exactly the same. However, the running time with SOS2 formulation increases. The computational time is provided in table 4.

Table 4 – Running time in seconds of two piecewise linear approximation formulations

Instance	Formulation with additional binary variables	SOS2 formulation
T3n10	11.01	21.95
T6n10	446.58	904.72

The maximum number of customers on each route also affects the results and the running time. The computational results of the instance for 3 periods, 10 customers, 5 breakpoints (for model 3 and 4) and the maximum number of customers equal to 2 and 3 are presented in table 5.

It can be noticed that when we increase the maximum number of customers per route from 2 to 3 the results of the models are improved. In model 1 the costs decrease and in models 2, 3 and 4 the profit increases. If we increase the maximum number of customers per route the number of possible routes will increase as well, that causes the increase in the computational time. We can assume that when the capacity of the vehicle is reached the increase of the maximum number of customers per route will not lead to any improvements of the results and will only increase the computational time.

Table 5 - Maximum number of customers per route

Instance	Criterion	Model 1		Model 2		Model 3		Model 4	
	maximum number of customers per route	2	3	2	3	2	3	2	3
T3n10	profit	8895.82	8360.44	17371.90	19672.90	30140.60	33666.60	22791.70	24474.20
	revenue	19030.40	17890.40	28250.40	31192.80	40945.20	44431.60	28250.40	31192.80
	costs	10134.59	9530.28	10878.52	11519.86	10804.52	10765.00	5458.68	6718.56
	transportation costs	3432.12	2817.40	4135.40	4763.37	3542.09	3420.96	4135.40	4763.37
	inventory holding costs at customers	288.43	289.24	488.68	613.80	729.89	835.13	488.68	613.80
	inventory holding costs at the supplier	1990.20	1999.80	1830.60	1718.85	2108.70	2085.07	705.00	668.25
	penalty	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
	production costs	4423.84	4423.84	4423.84	4423.84	4423.84	4423.84	129.60	673.14
	produced amount	1905	1905	1905	1905	1905	1905	58	308
	shipped amount of product	1116	1041	1641	1891	1192	1211.75	1641	1891
	consumed amount of product	1905	1905	1905	1905	1038	886.125	1905	1905
	profit per unit shipped	7.97	8.03	10.59	10.40	25.29	27.78	13.89	12.94
	time (seconds)	0.25	0.80	0.55	5.25	11.01	159.58	0.58	6.28

It is possible to calculate such non-linear functions as square of a variable using CPLEX solver. In model 4 for a competitive market we solved the non-linear function using piecewise linear approximation in order to make the model more general. However, in our case the function of the total production costs is a quadratic function of the production rate (Rs_t^2) and we can solve it without using approximation. The computational results of the instance for 6 periods and 10 customers with maximum 2 customers per route and 5 breakpoints for linear approximation are provided in table 6.

Table 6 - Quadratic function of the production rate

Instance	Criterion	Approximated Rs_t^2	Rs_t^2
T6n10	profit	45131.30	45134.50
	revenue	60032.00	60032.00
	costs	14900.71	10412.36
	transportation costs	8542.68	8542.68
	inventory holding costs at customers	865.73	865.73
	inventory holding costs at the supplier	1009.73	1003.95
	penalty	0.00	0.00
	production costs	4482.57	4485.10
	produced amount	1961	1961
	shipped amount of product	3544	3544
	consumed amount of product	3810	3810
	profit per unit shipped	12.73	12.74
	time (seconds)	0.70	1.48

There are insignificant differences in the results, however the computational time of solving a quadratic function is higher than the time that is needed to solve the piecewise linear approximation.

5.2.2. Economical tests

To understand the behavior of the models we run problems of 3 different sizes: small (3 periods and 5 customers (T3n5)), medium (3 periods and 10 customers (T3n10)) and large (6 periods and 10 customers (T6n10)). For all the instances the number of breakpoints (for model 3 and 4) is 5 and the maximum number of customers per route is 2. The computational results are demonstrated in Table 7 (a, b, c).

Table 7.a - Economic criteria (instance T3n5)

Instance	Criterion	Model 1	Model 2	Model 3	Model 4
T3n5	profit	3249.70	9989.05	12476.30	11394.30
	revenue	6644.00	14719.20	17038.50	14719.20
	costs	3394.30	4730.15	4562.18	3324.91
	transportation costs	1445.74	2791.26	2504.49	2791.26
	inventory holding costs at customers	85.79	194.32	206.27	187.88
	inventory holding costs at the supplier	639.90	521.70	608.15	200.25
	penalty	0.00	0.00	20.40	0.00
	production costs	1222.87	1222.87	1222.87	145.521
	produced amount	579	579	579	67
	shipped amount of product	262	577	473	577
	consumed amount of product	579	579	421	579
	profit per unit shipped	12.40	17.31	26.38	19.75

Table 7.b - Economic criteria (instance T3n10)

Instance	Criterion	Model 1	Model 2	Model 3	Model 4
T3n10	profit	8895.82	17371.90	30140.60	22791.70
	revenue	19030.40	28250.40	40945.20	28250.40
	costs	10134.59	10878.52	10804.52	5458.68
	transportation costs	3432.12	4135.40	3542.09	4135.40
	inventory holding costs at customers	288.43	488.68	729.89	488.68
	inventory holding costs at the supplier	1990.20	1830.60	2108.70	705.00
	penalty	0.00	0.00	0.00	0.00
	production costs	4423.84	4423.84	4423.84	129.60
	produced amount	1905	1905	1905	58
	shipped amount of product	1116	1641	1192	1641
	consumed amount of product	1905	1905	1038	1905
	profit per unit shipped	7.97	10.59	25.29	13.89

Table 7.c - Economic criteria (instance T6n10)

Instance	Criterion	Model 1	Model 2	Model 3	Model 4
T6n10	profit	27022.20	38023.50	52873.50	45131.30
	revenue	48360.80	60032.00	76495.80	60032.00
	costs	21338.57	22008.48	23622.25	14900.71
	transportation costs	7907.39	8542.68	9118.19	8542.68
	inventory holding costs at customers	746.51	865.73	1340.85	865.73
	inventory holding costs at the supplier	3837.00	3752.40	4313.63	1009.73
	penalty	0.00	0.00	1.91	0.00
	production costs	8847.67	8847.67	8847.67	4482.57
	produced amount	3810	3810	3810	1961
	shipped amount of product	2939	3544	2862	3544
	consumed amount of product	3810	3810	2860	3810
	profit per unit shipped	9.19	10.73	18.47	12.74

If we look at the numerical results of the three instances we can see that the models behave in a similar way. In model 1 we try to minimize costs, therefore, the optimal solution is to deliver only the amount of product that is needed to satisfy the demand in the considered time horizon or the additional amount that does not increase transportation and inventory holding costs. As a consequence, the inventory level at customers at the end of the planning horizon is minimized and tends to 0. This feature of the model can be considered as a drawback because customers will need to be served right after the considered horizon. Thus, the transportation costs will be considerably high in the first period of the next planning horizon. In opposite, in model 2 customers have a high inventory level at the end of the planning horizon, because the model tries to increase the revenue increasing the shipped amount of product. It means that we need to deliver less in the next planning horizon. In this case the inventory holding costs at customers and transportation costs increase, despite this the profit per unit increases as well. It means that the costs increase less than the revenue. So, we can say that we distribute the product in a more profitable way.

In model 3 for a monopoly the distributed and consumed amount of product is less than in other models because the model chooses higher unit sales revenue, which causes lower demand. Because of the lower consumption rate the inventory holding costs at the supplier and customers increase with the given production rate. However, the total profit and the profit per unit shipped increase. In this model we got the penalty for unsatisfied demand. It can be because the price limit is lower than the optimal one or because the model chooses the optimal combination of the price and corresponding demand even if we cannot completely satisfy the demand in some periods.

In model 4 for a competitive market the shipped amount of product is the same as in model 2. So, the inventory level at customers is high at the end of the planning horizon. However, the production costs and inventory holding costs at the supplier are much lower because in model 4 we can adjust the production rate and produce no more than necessary. Thus, the produced amount is lower and the supplier has zero inventory at the end of the planning horizon. It means that the supplier has to produce more in the next planning horizon. Though, high inventory level at customers gives the opportunity to the supplier to produce the product in the beginning of the next planning horizon. Also, it should be noticed that with the production rate chosen in model 4 the unit production costs are lower than the one in model 2 with the fixed production rate.

It is necessary to mention another feature of the models. In the first model with cost minimization we must satisfy all the demand. Therefore, if the inventory at the supplier or the fleet of vehicles is not enough or the capacity of the vehicles is not sufficient to serve all customers, the problem will be infeasible. However, models 2, 3 and 4 with profit maximization allow us to find the solution how to use limited available resources in a more profitable way.

6. Concluding remarks

In this master's thesis models for inventory routing problems with profit maximization as an objective function taking into account types of markets and corresponding ways of profit maximization were formulated. The way of linearization were found and the models were coded using AMPL. Finally, the generated instances were solved using CPLEX solver in order to test the models.

The solution for IRP with profit maximization provides decisions of the quantity to deliver to each customer and the vehicle routes for each discrete time period in order to increase the profit. In addition, the model for monopoly provides the possibility to adjust prices finding the optimal combination of price and demand and the model for perfectly competitive market allows to choose the optimal production rate with the optimal unit production costs that increases the profitability.

Developed models increase possibilities and can help companies to make better decisions taking into account more planning aspects at the same time. However, the topic has a potential for further research. The behavior of the models can be studied using long run simulation and improvements can be performed. Also, heuristics for larger sizes of instances can be developed.

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Appendix 1. Ampl code for Model 3 Inventory routing problems with profit maximization for monopoly

```
param T ;                   # number of discrete time periods
param n ;                   # set of customers
param SetSize ;             # set of routes
param SW ;                  # set of points for W1
param m >= 0 ;              #number of vehicles
param c {1..SetSize} >= 0 ;                 #cost of the routes
param a {1..n,1..SetSize} binary ;          # 1 if customer is served on route k
param U {1..n} >= 0 ;                       # maximum inventory level at customers
param h {1..n} >= 0 ;                       # inventory holding cost at customers
param Q >= 0 ;                              # vehicle capacity
param I0 {1..n} >= 0;                       # starting inventory level at customers
param rs >=0 ;                              # production rate of the supplier
param hs >= 0 ;                             # inventory holding cost at the supplier
param Us = 2 * sum {i in 1..n} U[i] ;       # maximum inventory level at the supplier
param B0 = sum {i in 1..n} U[i] ;           # starting inventory level at the supplier
param f {i in 1..n} >= 0  ;                 # penalty for unsatisfied demand
param ca;                                   #coefficient of rs^2 in a total cost function
param cb;                                   #coefficient of rs in a total cost function
param cd;                                   #constant in a total cost function
param M ;                                   # price limit
param lP {1..n} = 0 ;                       # lower bound for P
param lZ {1..n} = 0 ;                       # lower bound for Z
param uP {1..n} = M ;                       # upper bound for P
param uZ {i in 1..n} = T * U[i] ;           # upper bound for Z
param lW1 {i in 1..n} = (1/2) * (lP[i] + lZ[i]) ;       # lower bound for W1
param uW1 {i in 1..n} = (1/2) * (uP[i] + uZ[i]) ;       # upper bound for W1
param lW2 {i in 1..n} = (1/2) * (lP[i] - uZ[i]) ;       # lower bound for W2
param uW2 {i in 1..n} = (1/2) * (uP[i] - lZ[i]) ;       # upper bound for W2
param stepW1 {i in 1..n} = (uW1[i]-lW1[i])/(SW-1) ;  #the interval between breakpoints for W1
param stepW2 {i in 1..n} = (uW2[i]-lW2[i])/(SW-1) ; #the interval between breakpoints for W2
param w1 {1..n,1..SW} ;                                 # points for W1
param w2 {1..n,1..SW} ;                                 # points for W2
param g1 {i in 1..n,j in 1..SW} = w1[i,j]^2 ;           # w1^2
param g2 {i in 1..n,j in 1..SW} = w2[i,j]^2 ;           # w2^2
param e ;                                               # coefficient of P in the demand function
param d ;                                               # constant in the demand function

var I {1..n,0..T} >= 0 ;            # inventory level at i at time t (after consumption)
var Ship {1..n,1..SetSize,1..T} >= 0 ; #quantity shipped to i at time t by vehicle v using route k
var Y {1..SetSize,1..T} binary ;    #1 if route k is used at time t by vehicle v
var B {t in 0..T}>=0 ;              #inventory level at the supplier at time t
var C {1..n,1..T} >= 0 ;            #amount of product consumed at customers
var P {i in 1..n} >= lP[i], <= uP[i] ;                  # unit price for each customer
```

```
var Z {i in 1..n} >= lZ[i], <= uZ[i] ;                  # sum of Xikt on k and t
var W1 {i in 1..n} >= lW1[i], <= uW1[i];                #auxiliary variables
var W2 {i in 1..n} >= lW2[i], <= uW2[i] ;               #auxiliary variables
var lambda1 {1..n,1..SW} >= 0 ;                         #lambda for W1
var lambda2 {1..n,1..SW} >= 0 ;                         #lambda for W2
var Sb {1..n,1..SW-1} binary ;      # 1 if the interval between points of W1 is used, 0 otherwise

maximize Total_profit :
( sum {i in 1..n, j in 1..SW} g1[i,j] * lambda1[i,j] - sum {i in 1..n, j in 1..SW} g2[i,j] * lambda2[i,j] ) -
(sum {t in 1..T,k in 1..SetSize} c[k] * Y[k,t] + sum {i in 1..n,t in 1..T} h[i] * I[i,t] +
sum {t in 1..T} hs * B[t]) –
sum {i in 1..n,t in 1..T} f[i] * (e * P[i] + d - C[i,t]) - T * (ca * rs^2 + cb * rs + cd);

subject to Capacity {t in 1..T,k in 1..SetSize} : sum {i in 1..n} Ship[i,k,t] <= Q * Y[k,t] ;
#the quantity delivered by vehicle is not greater than its capacity
subject to Visit {t in 1..T,i in 1..n,k in 1..SetSize} : Ship[i,k,t] <= Q * a[i,k] * Y[k,t] ;
#delivery takes place only if a customer is visited with a route by the vehicle
subject to Vehicles {t in 1..T} : sum {k in 1..SetSize} Y[k,t] <= m ;
# the number of routes per day is limited by number of vehicles

subject to Initial_inv {i in 1..n} : I[i,0] = I0[i] ; # initial inventory at customers
subject to Inventory {i in 1..n,t in 1..T}: I[i,t] = I[i,t-1] + sum {k in 1..SetSize} Ship[i,k,t] - C[i,t] ;
#inventory level for each customer  in each time period
subject to Inv_capacity {i in 1..n,t in 1..T}:  sum {k in 1..SetSize} Ship[i,k,t] <= U[i] - I[i,t-1] ;
# inventory capacities at customers

subject to Consumption {i in 1..n,t in 1..T}:  C[i,t] <= e * P[i] + d ;
# consumed amount of product has to be less than or equal to the consumption rate

subject to Initial_inv_supplier: B[0] = B0 ; #initial inventory at the supplier
subject to Inventory_supplier {t in 1..T} : B[t] = B[t-1] + rs - sum {i in 1..n,k in 1..SetSize} Ship[i,k,t] ;
#inventory level at the supplier in each time period
subject to Inv_capacity_supplier {t in 1..T}: B[t] + rs <= Us ;  #inventory capacity at the supplier

subject to Z_variable {i in 1..n} : Z[i] = sum {k in 1..SetSize, t in 1..T} Ship[i,k,t];
#introducing a new variable Z which is equal to the sum of shipped amount of product over time
horizon and routes
subject to equalW1 { i in 1..n} : sum {j in 1..SW} w1 [i,j] * lambda1[i,j] = W1[i] ;
# the variable W1 is equal to the sum of breakpoints multiplied by lambda
subject to One1 {i in 1..n} : sum {j in 1..SW} lambda1[i,j] = 1 ;
#the sum of lambda must be equal to 1
subject to equalW2 { i in 1..n}: sum {j in 1..SW} w2 [i,j] * lambda2[i,j] = W2[i] ;
# the variable W2 is equal to the sum of breakpoints multiplied by lambda
subject to One2 { i in 1..n}: sum {j in 1..SW} lambda2[i,j] = 1 ;
#the sum of lambda must be equal to 1

subject to W_1 {i in 1..n} : W1[i] = 0.5 * (P[i] + Z[i]) ; #auxiliary variable W1
subject to W_2 {i in 1..n} : W2[i] = 0.5 * (P[i] - Z[i]) ; #auxiliary variable W2
```

```
subject to One3 {i in 1..n} : sum {j in 1..SW-1} Sb[i,j] = 1;  #only one interval can be chosen
subject to Lambda1_1 {i in 1..n}: lambda1[i,1] <= Sb[i,1] ;
#lambda[i,1] can be more than 0 only if the first interval is chosen
subject to Lambda1_s {i in 1..n, j in 2..SW-1}: lambda1[i,j] <= Sb[i,j-1] + Sb[i,j] ;
#lambda[i,j] can be more than 0 only if one of the intervals connected to the breakpoint is chosen
subject to Lambda1_SW {i in 1..n}: lambda1[i,SW] <= Sb[i,SW-1] ;
#lambda [i,SW] can be more than 0 only if the last interval is chosen

### Route generator###

minimize TotCost {b in HHH}: sum {(i,j) in LINKS[b]} cost[i,j] * X[i,j, b];

subj to Tour {b in HHH,  i in COMBS[b]}:
  sum {(i,j) in LINKS[b]} X[i,j,b] + sum {(j,i) in LINKS[b]} X[j,i, b] = 2;
subj to SubtourElim {b in HHH, k in MM[b] }:
   sum { i in SUB_CYCLE[k], j in COMBS[b] diff SUB_CYCLE[k]: (i,j) in LINKS[b]} X[i,j, b] +
   sum {i in SUB_CYCLE[k], j in COMBS[b] diff SUB_CYCLE[k]: (j,i) in LINKS[b]} X[j,i, b] >= 2;
#   These constraints say that the number of arcs in the solution that connect a
#   node in POW[k] to a node *not* in SUB_CYCLE[k] must be at least 2.
```

Appendix 2. Ampl code for Model 4 Inventory routing problems with profit maximization for perfect competition

```
param T ;                   # number of discrete time periods
param n ;                   # set of customers
param SetSize ;             # set of routes
param SRs ;                 # set of points for w
param m >= 0 ;              #number of vehicles
param e ;                   # coefficient of P in the demand function
param d ;                   # constant in the demand function
param c {1..SetSize} >= 0 ;         #cost of routes
param a {1..n,1..SetSize} binary ; # 1 if customer is served on route k
param rate {1..n} >= 0 ;            # quantity of product consumed by customer per unit of time
param U {1..n} >= 0 ;               # maximum inventory level at customers
param h {1..n} >= 0 ;               # inventory holding cost at customers
param Q = (1.5 * sum {i in 1..n} rate[i] ) / m ;      # vehicle capacity
param I0 {i in 1..n} = U[i] - rate[i] ;               # starting inventory level at customers
param hs >= 0 ;                                       # inventory holding cost at the supplier
param Us = 2 * sum {i in 1..n} U[i] ;                 # maximum inventory level at the supplier
param B0 = sum {i in 1..n} U[i] ;                     # starting inventory level at the supplier
param p {i in 1..n} = d/(-e) - rate[i] /(-e)  ;
# sales revenue per unit of product shipped to a customer
param f {i in 1..n} = 0.2 * p[i] ;    # penalty for unsatisfied demand
param ca;                             #coefficient of Rs^2 in a total cost function
param cb;                             #coefficient of Rs in a total cost function
param cd;                             #constand in a total cost function
param uRs {1..T} = sum {i in 1..n} rate[i] ;              # upper bound for Rs
param lRs {1..T} = 0 ;                                    # lower bound for Rs
param step {t in 1..T} = (uRs[t]-lRs[t])/(SRs-1) ;        # the interval between breakpoints
param w {1..T,1..SRs} ;                                   # points for Rs
param g {t in 1..T,j in 1..SRs} = ca * w[t,j]^2;          # ca * w^2

var I {1..n,0..T} >= 0 ;              # inventory level at i at time t (after consumption)
var Ship {1..n,1..SetSize,1..T} >= 0 ; #quantity shipped to i at time t by vehicle v using route k
var Y {1..SetSize,1..T} binary ;      #1 if route k is used at time t by vehicle v
var B {t in 0..T}>=0 ;                #inventory level at the supplier at time t
var C {1..n,1..T} >= 0 ;              #amount of product consumed at customers
var Rs {t in 1..T} >= lRs[t], <= uRs[t];      #production rate
var lambda {1..T,1..SRs} >=0 ;                #lambda for Rs

maximize Total_profit :
sum {i in 1..n,k in 1..SetSize,t in 1..T} p[i] * Ship[i,k,t] - (sum {t in 1..T,k in 1..SetSize} c[k] * Y[k,t] +
sum {i in 1..n,t in 1..T} h[i] * I[i,t] + sum {t in 1..T} hs * B[t]) -
sum {i in 1..n,t in 1..T} f[i] * (rate[i] - C[i,t]) -
```

```
( sum {t in 1..T,j in 1..SRs}  g[t,j] * lambda[t,j] + sum {t in 1..T} (cb * Rs[t] + cd)) ;

subject to Capacity {t in 1..T,k in 1..SetSize} : sum {i in 1..n} Ship[i,k,t] <= Q * Y[k,t] ;
#the quantity delivered by vehicle is not greater than its capacity
subject to Visit {t in 1..T,i in 1..n,k in 1..SetSize} : Ship[i,k,t] <= Q * a[i,k] * Y[k,t] ;
#delivery takes place only if a customer is visited with a route by the vehicle
subject to Vehicles {t in 1..T} : sum {k in 1..SetSize} Y[k,t] <= m ;
# the number of routes per day is limited by number of vehicles

subject to Initial_inv {i in 1..n} : I[i,0] = I0[i] ; # initial inventory at customers
subject to Inventory {i in 1..n,t in 1..T}: I[i,t] = I[i,t-1] + sum {k in 1..SetSize} Ship[i,k,t] - C[i,t] ;
#inventory level for each customer  in each time period
subject to Inv_capacity {i in 1..n,t in 1..T}:  sum {k in 1..SetSize} Ship[i,k,t] <= U[i] - I[i,t-1] ;
# inventory capacities at customers

subject to Consumption {i in 1..n,t in 1..T}:  C[i,t] <= rate[i] ;
# consumed amount of product has to be less than or equal to the consumption rate

subject to Initial_inv_supplier: B[0] = B0 ; #initial inventory at the supplier
subject to Inventory_supplier {t in 1..T} : B[t] = B[t-1] + Rs[t] - sum {i in 1..n,k in 1..SetSize}
Ship[i,k,t] ;  #inventory level at the supplier in each time period
subject to Inv_capacity_supplier {t in 1..T}: B[t-1] + Rs[t] <= Us ;  #inventory capacity at the
supplier

subject to equalRs {t in 1..T} : sum {j in 1..SRs} w [t,j] * lambda[t,j] = Rs[t] ;
#variable Rs is equal to the sum of breakpoints multiplied by lambda
subject to One {t in 1..T} : sum {j in 1..SRs} lambda[t,j] = 1 ; #the sum of lambda must be equal to 1

### Route generator###

minimize TotCost {b in HHH}: sum {(i,j) in LINKS[b]} cost[i,j] * X[i,j, b];

subj to Tour {b in HHH,  i in COMBS[b]}:
  sum {(i,j) in LINKS[b]} X[i,j,b] + sum {(j,i) in LINKS[b]} X[j,i, b] = 2;
subj to SubtourElim {b in HHH, k in MM[b] }:
   sum { i in SUB_CYCLE[k], j in COMBS[b] diff SUB_CYCLE[k]: (i,j) in LINKS[b]} X[i,j, b] +
   sum {i in SUB_CYCLE[k], j in COMBS[b] diff SUB_CYCLE[k]: (j,i) in LINKS[b]} X[j,i, b] >= 2;
#   These constraints say that the number of arcs in the solution that connect a
#   node in POW[k] to a node *not* in SUB_CYCLE[k] must be at least 2.
```