

Scientific Stochastic Volatility Models for the European Carbon Markets: forecasting and extracting conditional moments

Abstract

This paper builds and implements a multifactor stochastic volatility model for the latent (and observable) volatility of the carbon front December forward contracts at the European Carbon Exchanges, applying Bayesian Markov chain Monte Carlo simulation methodologies for estimation, inference, and model adequacy assessment. Stochastic volatility is the main way time-varying volatility is modelled in financial markets. Our main objective is therefore to structure a scientific model specifying volatility as having its own stochastic process. Appropriate model descriptions broaden the applications into derivative pricing purposes, risk assessment and asset allocation and portfolio management. From an estimated optimal and appropriate stochastic volatility model, the paper reports risk and portfolio measures, extracts conditional one-step-ahead moments (smoothing), forecast one-step-ahead conditional volatility (filtering), evaluates shocks from conditional variance functions, analyses multi-step-ahead dynamics, and calculates conditional persistence measures. The analysis adds insight and enables forecasts to be made, building up the methodology for developing valid scientific commodity market models.

Classification: C11, C63, G17, G32

Keywords: Stochastic Volatility, Bayesian Estimators, Metropolis-Hastings Algorithm, Markov Chain Monte Carlo (MCMC) Simulations, GSM-Projection-Reprojection

1 Introduction

This paper builds and assesses scientific stochastic volatility (SV) models for the European Carbon markets and traded among others at NASDAQ OMX Commodities Exchange, European Energy Exchange (EEX) and Intercontinental Commodity Exchange¹(ICE)². The European Commission established in 2005 the European Emission Trading Scheme (EU-ETS), a cap-and-trade scheme for emission allowances (EUA). In Europe, each country defines their total amount of emission allowance in National Allowance plans (NAP). Active trading in organized markets (tradable commodities) occurs when firms abate emissions and sell their allowances, while other firms require more allowances than allocated initially. Three phases has been announced; phase I running from 2005 to 2007, phase II running from 2008 to 2012, and phase III designed to run from 2013 to 2020. The price dynamics of EUAs in the new and immature Phase I market have been studied extensively (e.g. Hinterman (2010), Conrad et al. (2010)). Understanding the price dynamics in the more mature markets of Phase II and Phase III of EU-ETS is therefore more relevant for an efficient market design and carbon abatement costs learning. An actively traded global annual carbon market volume of € 92 billion (2010) is highly relevant for carbon traders, risk management and asset allocation³. In 2012 the Intercontinental Exchange had 91.6% of the total traded market and in January 2013 the exchange surpassed 40.000 (4000) EUA Futures ADV – 1.000 tCO₂e (EUA Options ADV – 1.000 tCO₂e) average daily volume. The ICE reported 115 ICE market Emissions members in January 2013. Knowledge of the empirical properties of the forward December carbon prices is important when constructing risk-assessment and management strategies. Market participants who understand the dynamic behaviour of carbon prices are more likely to have realistic expectations about future prices and the risks to which they are exposed.

Time-varying volatility is endemic in financial markets. Such risks may change through time in complicated ways, and it is natural to build stochastic models for the temporal evolution in volatility. The main objective of the paper is therefore to structure a scientific model specifying

volatility as having its own stochastic process, appropriately describing the evolution of the carbon market volatility. The implementation adapts the MCMC estimator proposed by Chernozhukov and Hong (2003), substantially superior to conventional derivative based hill climbing optimizers for this stochastic class of problems. Moreover, under correct specification of the structural model the normalized value of the objective function is asymptotically χ^2 distributed (and the degrees of freedom is well specified). An appropriate and well-specified SV model for the European carbon markets broaden applications into derivative pricing purposes, risk assessment and management, asset allocation and portfolio management.

Stochastic volatility models have an intuitive and simple structure and can explain the major stylized facts of asset, currency and commodity returns. The carbon price volatility investigation is important for several reasons. First, carbon price volatility could hinder investments in new and advanced technology and equipment. Second, if carbon prices and other energy and commodity prices are co-integrated, a volatile market might make it harder for consumers and businesses to predict their raw-material costs⁴. In a booming economy, an increase in prices may make it harder for economic growth to occur. Carbon prices move relatively slowly when conditions are calm, while they move faster when there is more news, uncertainty, and trading. One of the most prominent stylized facts of returns on financial assets is that their volatility changes over time. As the most important determinant of the price of an option is the uncertainty associated with the price of the underlying asset, volatility is of paramount importance in financial analysis. Risk managers are particularly interested in measuring and predicting volatility, as higher levels imply a higher chance of large adverse price changes. For commodity markets like other markets, the motivation for stochastic volatility is the observed non-constant and frequently changing volatility. The SV implementation is an attempt to specify how the volatility changes over time. Bearing in mind that volatility is a non-traded instrument, which suggests imperfect estimates, the volatility can be interpreted as a latent variable that can be modelled and predicted through its direct influence on the magnitude of returns. Besides, as the European market writes options on

the carbon December future contracts, SV models are also motivated by the natural pricing of these options, when over time volatility change. Finally, for the carbon markets observed returns and volatility changes seem so frequent that it is appropriate to model both returns and volatility by random variables.

The paper focuses on the Bayesian Markov Chain Monte Carlo (MCMC) modelling strategy used by Gallant and McCulloch (2011) and Gallant and Tauchen (2010a, 2010b)⁵ implementing uni- and multivariate statistical models derived from scientific considerations. The method is a systematic approach to generate moment conditions for the generalized method of moments (GMM) estimator (Hansen, 1982) of the parameters of a structural model. Moreover, the implemented Chernozhukov and Hong (2003) estimator keeps model parameters in the region where predicted shares are positive for every observed price/expenditure vector. For conventional derivative based hill climbing algorithms this is nearly impossible to achieve. Moreover, the methodology supports restrictions, inequality restrictions, and informative prior information (on model parameters and functionals of the model). Asset pricing models as the habit persistence model of Campbell and Cochrane (1999), the long run risk model of Bansal and Yaron (2004) and the prospect theory model of Barberis, Huang, and Santos (2001) are all implemented. For the SV model implementation, the enhanced statistical and scientific stochastic model calibration methodologies can greatly enhance portfolio management, elaborate and extend the decomposition and aggregation of overall corporate and institutional risk assessment and management. In fact, appropriate MCMC estimated SV model simulations can generate probability distributions for the calculation of value at risk (VaR/CVaR) and Greek letters for portfolio rebalancing and model parameters can be the basis for forecasting the mean and volatility for forward assessment of risk, portfolio management, and other derivative pricing purposes. However, on the downside, as volatility is latent (and unobservable) coinciding with the fact that the conditional variances are complex functions complicating the maximum likelihood, estimations will be imperfect and a single optimal estimation technique is probably not available.

It is simple to make forecasts using the MCMC framework. Hence, both the mean and volatility are able to be forecasted adding content to future contract prices. Moreover, the re-projection method (Gallant and Tauchen, 1998) based on long simulated data series can extend projections. The post-estimation analysis can be used as a general-purpose technique for characterizing the dynamic response of the partially observed system to its observable history. Forecasting the conditional moments and the use of filtered volatility (with a purely ARCH-type meaning) and multistep-ahead dynamics are some features that really add strength to the methodology building of scientifically valid models. Ultimately, the analysis may therefore contribute to more realistic risk methodologies for energy markets and for market participants an improved understanding of the general stochastic behaviour inducing more realistic expectations about future prices and the risks to which they are exposed. The relatively new research called carbon finance, analyse the differences in EUA price dynamics between periods by considering jumps and spikes as well as phases of high volatility, volatility clustering and heteroscedasticity. The research confirms the presence of the stylized facts like skewness, excess kurtosis and different phases of volatility behaviour (e.g., Paoletta and Taschini (2008), Benz and Trück (2009), and Conrad et al. (2010). All articles suggest the use of (G)ARCH-type models for the volatility equation (obtain heteroscedasticity consistent co-variances). They also show that the influence of fundamentals can be included in the mean equation. This paper is the first paper to use random variables and therefore stochastic volatility for the carbon prices. Moreover, previous research on other energy market prices is voluminous. Studies adopting the Heath-Jarrow-Morton (HJM) assuming a dynamics for the forward and swap price evolution have been suggested. In particular, Bjerksund et al. (2000), Keppo et al. (2004), Benth and Koekebakker (2008) and Kiesel et al. (2009) have used contracts for the NASDAQ OMX and EEX electricity markets. The same approach for energy markets in general can be found in Clewlow and Strickland (2000). However, modelling the price dynamics, where the contracts delivers over a period, creates challenges that are not present in the fixed income market theory (see Musiela and Rutkowski, 1998). To resolve this

problem the LIBOR models in interest rate theory (see Brigo and Mercurio, 2001) exclusively model contracts that are traded in the market and do not have delivery periods, which cannot be decomposed into other traded contracts. The result is a much more freedom to state reasonable stochastic dynamical models⁶ (Benth et al., 2008). Kiesel et al. (2009) proposes a two-factor model for electricity prices. Even though the research on the energy market is voluminous the LIBOR approach and modelling different parts of the term structure individually make the SV modelling interesting for derivative purposes, risk management and asset allocation. The contribution of our carbon study is therefore threefold.

First, this paper is one of the first scientific model investigation of the behaviour of the front December forward contracts traded at the European carbon emission exchanges. In particular, it provides the first study to our knowledge that examines modelling the return volatility using scientific stochastic volatility models together with the Bayesian estimation and inference methodology (MCMC). Second, it improves assessment methodology for the evaluation of model fit and empirical scientific content. Plotting suitable measures of location and scale of the posterior distributions assesses the scientific model. That is, plots showing small location changes and increasing scale is favourable for scientific well-defined models. Third, to our knowledge this paper is quite unique analysing conditional moment forecasts for the carbon market contracts (point estimates and densities). In particular, it investigates the possibility that lagged returns contribute little if any additional information about future returns and variances. Fourth, risk management (VaR/CVaR) and asset allocation (Greeks) measures are available for both conditional and unconditional moments. Finally, particle filtering, multistep-ahead forecast and persistence is reported for the contracts. The rest of the paper is therefore as follows. Section 2 presents background research and defines the data set. Section 3 describes the SV methodology. Section 4 conducts the GSM stochastic volatility specifications and assesses model validity. Section 5 interprets the model result. Section 6 uses the estimated model results for mean and volatility prediction, volatility filtering and describes how to extend the model assessment features

of the methodology for option pricing and implied volatilities calculations. Section 7 summarizes and concludes.

2 Background and Data set

The EU legal framework constitutes the basis for trading in EUAs. The EU Emission Trading Scheme is the main policy being introduced across Europe to handle emissions of carbon dioxide and other greenhouse gases, in order to counter the threat of climate change. The carbon trading is related to the fact that electricity generators fuelled by coal or natural gas emit substantial volumes of carbon dioxide. The emitters must now pay for these emissions and carbon contracts allow management of allowance price risks. To accomplish the needs a marketplace for European Union Allowances (EUAs) was established. All participants on the exchange can trade via open and cost-effective electronic access on equal terms.

The derivatives markets was first started in 2005 (February). The market is an electronic order book where participants can see the system orders (anonymously) and the best ask and bid prices with corresponding volumes. Pricing is established at the end of the opening phase and during continuous trading. The main carbon products traded are forward/futures and options. Similar to other international commodity markets, the majority of members on the derivative market use the market for risk management purposes. The major difference between the financial electricity market and the emissions market is the physical delivery of EUAs and CERs to the buyer and a financial settlement to the seller, while the financial electricity market has only financial settlement. Hence, the front year December future contract comprises physical delivery where the seller transfers EUAs to the buyer and the buyer makes financial settlement with the seller. In January 2013, the ICE market listed 22 EUA futures contracts with immediate physical delivery ranging from March 2013 to December 2020. For options, up to 16 contract months are listed on a quarterly expiry (March, Jun, September, and December), with 3 new contract months listed on expiry of the December contract. For all options the underlying contract is the December Future

of the relevant year. Quotations are all in euro, with a minimum contract size of 1 k tonne CO₂. Trading is available on a continuous trading platform as well as by voice execution by the marketplace desk using the euro. The daily closing prices are set at the end of each day. The prices are primarily influenced by fundamental factors known from the electricity market like weather, coal, natural gas and oil prices. The total amount of allocated allowances (the EU cap) is also important for the prices. The market facilitates an efficient, transparent and confidence-inspiring development of the emission market similar to the one for power. This is administered by strict requirements for information management in a regulated marketplace with standardized products.

The daily analyses cover the period from the end of 2007 until the start of 2013 (March 1st), a total of five consecutive years and approximately 1302 price change observations. Any signs of successful SV-model implementations for the forward market will indicate non-predictive market features and a minimum of weak-form market efficiency. Consequently, the carbon markets are applicable for enhanced risk management activities including pricing of hedging instruments as well as conventional portfolio/fund management procedures. For all market participants an efficient market suggests pricing mechanisms reflecting all relevant historical information indicating a foundation of non-predictive and efficient market pricing, which are important ingredients for successful market implementation. In the long run, only an effective market can give guarantees for market prices close to the marginal cost of expanding capacity. It is important that participants can infer that all hedging instruments both for short and long run are fairly priced at all times.

{Insert Table 1 about here}

{Insert Figure 1 about here}

The daily percentage change (logarithmic) of the data sets from the end of 2007 to the start of 2013 is $y_t, t = 1, \dots, 1302$. Characteristics of this (commodity) markets financially traded contracts are reported in Table 1. The mean is negative and the standard deviation, the maximum

value and the kurtosis are relatively high. Serial correlation in the mean equation is not strong and the Ljung-Box Q-statistic (1978) is only marginally significant. Volatility clustering using the Ljung-Box test statistic (1978) for squared returns (Q^2) and ARCH statistics is significantly present. The *KPSS* (Kwiatowski et al., 1992) statistic cannot reject neither level nor trend (12) stationary price change series. The Dickey-Fuller test adds support to stationary series. The *BDS* (Brock et al., 1999) test statistics report highly significant data dependence. The price level (a) and price change (log returns) (b) data series are shown in the upper part of Figure 1. From the return plots the series show some large returns at lower price levels towards the end of the series and the level of volatility seems to change randomly. The plots together with the ARCH- and Ljung-Box test statistics (Q^2) in Table 1 manifest volatility clustering. Moreover, the distributional and QQ-plots in parts (c) – (e) of Figure 1 show that the series distribution is more peaked and have fatter tails than a corresponding normal distribution. For the left and the right tails of the distribution, the number of large negative price changes seems to occur more often than large positive. However, the size of the returns is much higher for positive price changes. That is – the series will most likely have a negative drift with larger positive than negative price jumps. These facts together with a third moment different from zero and a fourth moment higher than three, indicates a non-normal distribution and leptokurtosis. Finally for later comparisons, the Value at Risk (VaR) and expected shortfall (CVaR) numbers report percentile and expected shortfall numbers for long positions at less than 2.5%. For our scientific model calibrations, these features found in the commodity series suggests nonlinear models, simply because linear models would not be able to generate these data.

3 Theoretical SV model Background and Motivation

Stochastic Volatility models provide alternative models and methodologies to (G)ARCH models⁷. The time-varying volatility is endemic in global financial markets. The SV approach specifies the predictive distribution of returns indirectly, via the structure of the model, rather than directly (ARCH). The main advantage of direct volatility modelling is convenience and perhaps more

natural. Relative to ARCH the SV model therefore has its own stochastic process without worries about the implied one-step-ahead distribution of returns recorded over an arbitrary time interval convenient for the econometrician. Moreover, simulation strategies developed to efficiently estimate SV models, have given access to a broad range of fully parametric models. This enriched literature has brought us closer to the empirical realities of the global markets⁸. However, SV and ARCH models explain the same stylized facts and have many similarities.

The starting point is the application of Andersen et al. (2002) considering the familiar stochastic volatility diffusion for an observed stock price S_t given by $\frac{dS_t}{S_t} = (\mu + cV_t) dt + \sqrt{V_t} dW_t$, where the unobserved volatility process V_t is either log linear or square root (affine) (W_{1t}). The W_{1t} and W_{2t} are standard Brownian motions that are possibly correlated with $\text{corr}(dW_{1t}, dW_{2t}) = \rho$. Andersen et al. (2002) estimate both versions of the stochastic volatility model with daily S&P500 stock index data, 1953-December 31, 1996. Both SV model versions are sharply rejected. However, adding a jump component to a basic SV model improves the fit sharply, reflecting two familiar characteristics: thick non-Gaussian tails and persistent time-varying volatility. A SV model with two stochastic volatility factors show encouraging results in Chernov et al. (2003). The authors consider two broad classes of setups for the volatility index functions and factor dynamics: an affine setup and a logarithmic setup. The models are estimated using daily data on the DOW Index, January 2, 1953 – July 16, 1999. They find that models with two volatility factors do much better than do models with only a single volatility factor. They also find that the logarithmic two-volatility factor models outperform affine jump diffusion models and basically provide acceptable fit to the data. One of the volatility factors is extremely persistent and the other strongly mean-reverting. The SV model for the European energy market applies the logarithmic model with two stochastic volatility factors (Chernov et al., 2003). Moreover, the SV model is extended to facilitate correlation between the mean (W_{1t}) and the two stochastic volatility factors (W_{2t}, W_{3t}). The main argument for the correlation modelling is to introduce asymmetry effects (correlation

between return innovations and volatility innovations). This paper formulation of a SV model for the European energy market's price change process (y_t) therefore becomes

$$\begin{aligned}
 y_t &= a_0 + a_1 (y_{t-1} - a_0) + \exp(V_{1t} + V_{2t}) \times u_{1t} \\
 V_{1t} &= b_0 + b_1 (V_{1,t-1} - b_0) + u_{2t} \\
 V_{2t} &= c_0 + c_1 (V_{2,t-1} - c_0) + u_{3t} \\
 u_{1t} &= W_{1t} \\
 u_{2t} &= s_1 \left(r_1 \times W_{1t} + \sqrt{1 - r_1^2} \times W_{2t} \right) \\
 u_{3t} &= s_2 \left(r_2 \times W_{1t} + \left((r_3 - (r_2 \times r_1)) / \sqrt{1 - r_1^2} \right) \times W_{2t} + \sqrt{1 - r_2^2 - \left((r_3 - (r_2 \times r_1)) / \sqrt{1 - r_1^2} \right)^2} \times W_{3t} \right)
 \end{aligned} \tag{1}$$

where W_{it} , $i = 1, 2$ and 3 are standard Brownian motions (random variables). The parameter vector is $\rho = (a_0, a_1, b_0, b_1, s_1, c_0, c_1, s_2, r_1, r_2, r_3)$. The r 's are correlation coefficients from a Cholesky decomposition; enforcing an internally consistent variance/covariance matrix. Early references are Rosenberg (1972), Clark (1973) and Taylor (1982) and Tauchen and Pitts (1983). More recent references are Gallant et al. (1991, 1997), Andersen (1994), Durham (2003), Shephard (2004), and Chernov et al. (2003). The model above has three stochastic factors. Extensions to four and more factors can be easily implemented through this model setup. Even jumps with the use of Poisson distributions for jump intensity are applicable (complicates estimations considerably). As shown by Chernov et al. (2003) liquid financial markets make a much better model fit introducing two stochastic volatility factors. One of the volatility factors that is strongly mean reverting to fatten tails, a second factor that is extremely persistent, to capture volatility clustering.

The paper implements a computational methodology proposed by Gallant and McCulloch (2011) and Gallant and Tauchen (2010a, 2010b) for statistical analysis of a stochastic volatility model derived from a scientific process⁹. The scientific stochastic volatility model cannot generate likelihoods but it can be easily simulated. SV models contain prior information but are only expressed in terms of model functionality that is not easily converted into an analytic prior on the

parameters but can be computed from a simulation. Intuitively, the approach may be explained as follows. First, a reduced-form auxiliary model is estimated to have a tractable likelihood function (generous parameterization). The estimated set of score moment functions encodes important information regarding the probabilistic structure of the raw data sample. Second, a long sample is simulated from the continuous time SV model. Using the Metropolis-Hastings algorithm and parallel computing, parameters are varied in order to produce the best possible fit to the quasi-score moment functions evaluated on the simulated data. In fact, if the underlying SV model is correctly specified, it should be able to reproduce the main features of the auxiliary score functions. An extensive set of model diagnostics and an explicit metric for measuring the extent of SV model failure are useful side-products. Finally, the third step is the re-projection method. The task of forecasting volatility conditional on the past observed data (akin to filtering in MCMC) or extracting volatility given the full data series (akin to smoothing in MCMC) may now be undertaken. Moreover, the post estimation analysis make an assessment of model adequacy possible by inferring how the marginal posterior distributions of a parameter or functional of the statistical model changes. The parameter maps of both models should correspond to the same data generating process and the statistical model should therefore also be identified by simulation from the scientific model.

4 The General Scientific Model methodology

The $y_t, t = 1, \dots, 1302$ is the percentage change (logarithmic) over a short time interval (day) of the price of a financial asset traded on an active speculative market. The methodology is used to estimate stochastic volatility models for the front December series. The first step for implementing the methodology is the moment generator. The projection method, described in Gallant and Tauchen (1992), provides an appropriate and detailed statistical description of the series. Starting from a VAR model, the methodology if necessary, elaborates the description of the data set from VAR, to Normal (G)ARCH, to Semi-parametric GARCH, and to Non-linear Nonparametric. Applying the BIC (Schwarz, 1976) values for model selection, the preferred

model for the data set is a semi-parametric GARCH model with 4 hermite polynomials for non-normal features of the series. The model is an AR(1) model for $\{y_t\}$ with a GARCH(1,1) conditional scale function and time homogeneous nonparametric innovation density inducing tails. Note that the dependence on the past is through the linear location function and the GARCH scale functions. Also note, that our statistical methodology describes the GARCH process using a *BEKK* (Engle and Kroner, 1995) formulation for the conditional variance allowing for *BIC*-efficient volatility asymmetry and level effects. For the contracts the level effects are insignificant and excluded from the statistical model. However, asymmetric volatility is included. Finally, the eigenvalue of variance function P & Q companion matrix is 0.959 and the eigenvalue of mean function companion matrix is 0.033. Finally, for evaluation purposes, the intermediate model residuals are exposed to elaborate statistical specification tests. The specification tests are shown in Table 2 for the optimal semi-parametric GARCH models. The test statistics suggest no data dependence, normal distributions and no volatility clustering. Hence, model misspecifications seem minimized and the series can be used for descriptive purposes of the future contracts. Some characteristics of the projected time series from the BIC calibration are reported in Figure 2. The conditional volatility together with a moving average (m -lags) of the squared residuals of an AR(1) regression model of the returns are reported in plot (a). The projected volatility seems not to be a reasonable compromise between $m=4$ and $m=15$. The volatility seems to change randomly. The one-step-ahead density $f_K(\tilde{y}_t | x_{t-1}, \hat{\theta})$, conditional on the values for $x_{t-1} = (\tilde{y}'_{t-L}, \dots, \tilde{y}'_{t-2}, \tilde{y}'_{t-1})'$, is plotted in (b). All lags are set at the unconditional mean of the data. The plots, peaked with fatter tails than the normal with some asymmetry, suggest only small non-normal features, typically shaped for data from a financial market. The features suggest well-behaved time series for the contracts. Moreover, the information contained in the plots for the mean and the volatility will be very useful for the implementation of the scientific SV models.

{Insert Table 2 about here}

{Insert Figure 2 about here}

The SV model implementation established a mapping between the statistical model and scientific model. The adjustment for actual number of observations and number of simulation must be carefully logged for final model assessment. Procedures applying an optimization routine together with an associated iterative run for model assessments, establish the reporting foundation and empirical findings from the Bayesian MCMC estimation that are given below. The optimal SV model from the parallel run model is reported in Table 3. The mode, mean and standard deviation are reported. The accompanying statistical model parameters are reported to the right (rescaled). The optimal Bayesian log posterior values are -1397.8 for the future carbon contracts. The statistical models in Table 3 seem to give roughly the same results as for the originally statistically estimated semi-parametric GARCH model. The statistical models suggest non-normal return distributions, positive drift, and serial-correlation in the mean and volatility equations. The presumption of an appropriate Geometric Brownian Motion (GBM) description for the carbon market therefore seems unrealistic.

{Insert Table 3 about here}

4.1 Model Evaluation and Parameter/Functional Assessments

The starting point is imposing the belief that the scientific model holds exactly. The idea is captured by recasting the problem so that the estimated parameters (η) of the statistical model are viewed as the parameter space of interest and constructs a prior that expresses a preference. We impose a single parameter (κ) to control prior beliefs about how close the parameters of the statistical model (η) should be to the manifold. The smaller κ is, the more weight is placed on η close to the manifold. The scientific model is therefore assessed using the marginal posterior distribution of interpretable features of the statistical model, by changing κ . If changing our prior beliefs so as to support η farther from the manifold results in location shifts of the posteriors that are appreciable from a practical point of view, we then conclude that the evidence in the likelihood is against the restriction corresponding to the scientific model. For the SV model we examine the low dimensional marginals of interest (θ). The assessment report uses the η - and θ -

parameter frequency distributions for $\kappa = 1, 10, 20$ and 100 distance from the manifold (M). For two η -parameters and the carbon contracts the results are reported in Figure 3. As can be observed from the plots, the distributions become wider when κ is increased but the mean (the location measure) does not seem to move significantly; suggesting a well-fitting scientific model. The figures report location and scale measures of the posterior distributions of η_4 and η_8 . All the parameters show close to negligible effects for the series¹⁰. The main effect of imposing the stochastic volatility model on the statistical model is to force symmetry on the conditional density. The η 's posterior densities show small changes by imposing priors with $\kappa = 1, 10, 20$, and 100. The chi-square statistics with 2 (ltheta-1-lrho) degrees of freedom are -3.403 with associated p -values of 0.182 (see Table 3). The p -values indicate a successful model fit. Finally, the normalized mean score vectors along with unadjusted standard errors, report the quasi t -statistics (not reported). All the quasi t -statistics are well below 1.0 indicating a successful fit for all mean score moments.

{Insert Figure 3 about here}

5 Empirical Findings and Portfolio and Risk Measures (post-estimation analysis)

Table 3 reports the θ -parameters mode and mean with associated standard deviations for the Carbon SV estimation applying the GSM procedure. Confidence intervals for the seven coefficients are calculated by inverting the criterion difference test based on the asymptotic chi-square (χ^2) distribution of the optimized objective function (Gallant et al., 1997). These criterion difference intervals reflect asymmetries in the objective function and are also preferred from a numerical analysis point of view. The results for the θ -parameter are summarized in Table 4. The confidence intervals are narrower and asymmetric relative to classical (i.e. Hessian) standard errors¹¹. Figure 4 reports a sub-sample of the scientific model kernel distributions for the θ_3 and θ_6 coefficients (a)-(b), respectively. Figure 4 (c) also reports the sci-mod-posterior log-likelihood graphically (the optimal sci-mod-posterior value in Table 3 should be found along this path). It appears that all parameter chains, distributions and the log-likelihood have found their mode and

look satisfactory (no large and persistent deviations away from the mean value). The SV model parameters show a negative drift and a positive serial correlation in the mean equations.

{Insert Table 4 about here}

{Insert Figure 4 about here}

For our period the contracts report an insignificant drift parameter of -0.034 (mode). Daily serial correlation is 0.033 with a standard error of 0.0196.. The contract does not show a high positive daily serial correlation in the mean and suggest non-predictability from serial correlation (significant serial correlation *at several lags* is needed to make the contract inefficiently priced). The conditional volatility parameters report a high and positive constant parameters (b_0), inducing relatively high unconditional volatility for the contracts. Specifically, the volatility equations have a constant clearly above one for the carbon contracts ($e^{0.792}$). The volatility shows a quite high persistence coefficient (b_1), with a mode of 0.985 (persistence). Importantly, the model implementation suggests a specification need for two instantaneous volatility factors. The contracts report a mean instantaneous volatility for s_1 of 0.076 and s_2 of 0.232. For the carbon markets we therefore find a daily mean volatility of 2.51% (39.66% per annum). For general financial markets these volatility figures can be considered relatively high. A mean daily drift of -0.034% (-8.5% per year) may signal a clearly negative trend in the market. The trend may follow the current negative trend in European economy in general. The asymmetric volatility coefficient (r_1) is significant and negative for the contracts (-0.52). The negative asymmetric coefficient suggests that the contracts show higher volatility from large price decreases.

The three-equation SV model can now be easily simulated at any length. The series features of the mean and volatility equations from a functional simulation (250 k) of the market are reported in Figure 5. The plots report a full- (250 k) and sub-sample (0.5 k) of the mean (a) and exponential volatility (b) equations. The plots (c) and (d) report features for the two instantaneous volatility factors (full-/sub-samples). The volatility factors seem to model two different flows of information to the market. One factor moves slowly while the second factor moves clearly

narrower but faster; that is – one slowly mean reverting factor provide volatility persistence and one rapidly mean reverting factor provides for the tails (see Chernov et al. (2003)). The plots in (e) and (f) report the mean and volatility densities and *QQ*-plots. The density distributions and the *QQ*-plots from a long run SV simulation give some extra insight to both the mean and volatility of the systems under consideration. The volatility density is close to normal and the exponential version is close to log-normally distributed. In contrast, the mean shows a negative skew of -0.43 and the kurtosis is 5.08. Hence, the contracts show existence of leptokurtosis in the mean data generating process, with too many observations around the mean and in the both tails and too few observations around one standard deviation from the mean, seem present. These results are consistent with the SV model coefficients results.

{Insert Figure 5 about here}

The SV-model estimation and inference give us also immediate access to Value at Risk (VaR/CVaR), and *Greek* Letters (with a quoted exercise price). As all measures are accessible for every stochastic run they will be available for reporting in distributional forms. A first run estimate of 99% VaR (CVaR) is 8.6% (11.4%). VaR and CVaR are normally best applied using extreme value theory (EVT) for smoothing out the tail results. Applying the estimated SV model for 10 *k* iterations and 1 million euro invested in the European carbon contracts, a maximum likelihood optimization of 99.9%, 99.5%, 99% and 97.5% VaR (a) and CVaR (b) calculations are reported in Figure 6. The tails of these distributions are of interest for risk managers engaged in this commodity market. The value of the one-day 99.9% VaR (CVaR) for a €1 million portfolio in the carbon markets is €1 million x 0.14763 = €147.630 (€1 million x 0.18187 = €181.870). More generally, our estimate of the one-day 99.9% VaR (CVaR) for a portfolio invested in the European carbon market is 14.8% (18.2%) of the portfolio value. For portfolio and asset allocation purposes the *Greek* letters delta, rho and theta are reported in Figure 7 (only deltas are reported). The distribution for an at-the-money (ATM) call (a) (put (b)) delta measure shows a mean of 0.497 (0.492) with an associated standard deviation of 0.0022 (0.0021). A credible ATM call option measure for the contracts seems therefore to have a 95% confidence interval ranging

from 0.493 to 0.502. However, we can do better than this simple unconditional forecasting procedure. We will now apply the third and final step in Gallant and Tauchen (1998), the re-projection step for post-estimation analysis and forecasting, which brings the real strengths to the methodology in building scientific valid models.

{Insert Figure 6 about here}

{Insert Figure 7 about here}

6 Forecasting and Extracting Volatility for the Carbon Markets

The re-projection methodology gets a representation of the observed process in terms of observables that incorporate the dynamics implied by the non-linear system under consideration. This post estimation analysis of the simulations entails mean and volatility predictions, filtering and model assessment. Scientific valid models cannot be built without enhanced assessment of model adequacy. Having the SV estimates of system parameters for our models, we simulate a very long realization of the state vector (250 k). Working within this simulation, we can calibrate the functional form of the conditional distributions. To approximate the SV model result using the score generator (f_K) values, it is natural to reuse the values of the original raw data calibration. The dynamics of the first two one-step-ahead conditional moments may contain important information for all market participants. Figure 8 shows the first moment $E[y_0|x_{-1}]$ paths, densities and QQ-plots in plots (a), (b) and (c); the second moment $Var[y_0|x_{-1}]$ paths, densities and QQ-plots in plots (d), (e) and (f). The first moment information conditional on all historical available data shows a one-day-ahead density (approximately from -1.25% to +1.0%). This is informative for daily risk assessment and management. Thus, using the whole history of observed data series implies a much narrower mean density indicating some relevant information from the history of the time series. One-step-ahead VaR and CVaR can be calculated using density percentiles from this conditional mean distribution. For the conditional mean the VaR (CVaR) is 0.33% (0.413%) and 0.461% (0.558%) for 2.5% and 0.5% percentiles, respectively. Moreover, repeating this procedure (tedious and time-consuming) calculating VaR/CVaR and Greeks for every run, we will also be able to report both the VaR/CVaR and the Greeks using credible density forecasts (not reported).

{Insert Figure 8 about here}

The explicit variance and standard deviation distributions are interesting for several applications with a special emphasis on derivative computations. However, the volatility does not change much from the original simulated SV model. The volatility is assumed latent and stochastic. The filtered volatility, which is the one-step-ahead conditional standard deviation evaluated at data values (x_{t-1}), may give us some extra information. The filtered volatility is a result of the score generator (f_k) and therefore volatility with a purely ARCH-type meaning. Alternatively, a Gauss-Hermite quadrature rule can be used. Figure 9 report a representation of the filtered volatility at the mean of the data series for the contracts (a). The densities display the typical shape for data from a financial market: peaked with fatter tails than the normal with some asymmetry. In Figure 9 and plot (b), we have plotted the distributions for several data values (x_{t-1}) from -10% to +10%. Interestingly, the largest values of x_{t-1} have the widest densities. The one-step-ahead filtered volatility seems therefore to contain more information than the SV-model, giving us conditional one-step-ahead densities. Based on the observation at day t it is therefore of interest to use the one-step-ahead standard deviation for several applications. This filtered volatility and the Gauss-Hermite quadrature shown in plot (c) can be used for one-step-ahead price calculations of any derivative. Figure 9 plot (d) reports the conditional variance function. The function reports the one-step-ahead dynamics of the conditional variance plotted against percentage growth (δ). Here we can interpret the graph as representing the consequences of a shock to the system that comes as a surprise to the economic agents involved. Asymmetry (“leverage effect”) can be evaluated from this function. From the (shock (δ)) plots for the contracts we see that the responses from negative (positive) shocks are higher than from positive (negative). The SV model negative ρ -parameter with a mode of -0.52 and a standard error of 0.122 signals quite a high negative mean and volatility correlation producing negative asymmetry. The sign of the parameter suggest larger volatility from negative returns.

{Insert Figure 9 about here}

Figure 9 plot (e) reports multistep ahead dynamics. The conditional volatility plots reveal the future dynamic response of system forecasts to contemporaneous shocks to the system. These generally nonlinear profiles can differ markedly when the sign of δ changes. However, the dynamics seems symmetric ($+/-\delta$) and higher shocks are mean reverting, while periods of low volatility induce higher volatility to come. Finally, Figure 9 plot (f) reports profile bundles for the volatility to assess persistence (similar plot for the mean is not reported). The profile bundles are over plots conditional to each observed datum¹². If the thickness of the profiles bundle tends to collapse to zero rapidly, the process reverts to its mean. If the thickness tends to retain its width, then the process is persistent. For the carbon volatility, the calculation of half-life (regression on margins) reveals approximately 24.53 with associated standard error of 0.27 days. All these findings from the post estimation analysis add more insight to building scientifically valid models. Based on these findings the post estimation analysis seems to add insight to building scientifically valid models for European carbon markets.

Finally, the most predominant applications for the SV model and re-projection is option pricing and implied volatility calculations for market risk premiums and pricing errors. From the previous section we can obtain asset prices $S_{i,T}$ at time T from simulations labelled by $i = 1, 2, \dots, N$. The

fair price for a call is generally $c = e^{-rT} E^Q [\max(S_T - X, 0)] = e^{-rT} \int_X^\infty (x - X) f_Q(x) dx$ and would be

estimated by $c_N = (e^{-rT} / N) \sum_{i=1}^N \max(S_{i,T} - X, 0)$, where X is the strike price. Using the long simulation of

$\{y_t^*, x_t^*\}$ from our optimal SV structural model and performing a projection to get

$\int y^* \hat{f}_K(y^* | x^*) dy^*$, where y^* is unobserved (volatility) and x^* observed (returns) variables,

establishes the needed and highly valuable re-projected volatility. Now, using the optimal SV model from the estimation date and risk neutral valuation (martingales), we can calculate derivative prices at any maturity and complexity. The methodology for pricing options and

implied volatility calculations using re-projected volatility is available for the European carbon market¹³.

7 Summary and Conclusions

This paper has used the Bayesian M-H estimator and GSM for a stochastic volatility representation for the European financial energy market contracts. The methodology is based on the simple rule: compute the conditional distribution of unobserved variables given observed data. The observables are the asset prices and the un-observables are a parameter vector, and latent variables. The inference problem is solved by the posterior distribution. Based on the Clifford-Hammersley (1970) theorem, $p(\theta, x/y)$ is completely characterized by $p(\theta/x, y)$ and $p(x/\theta, y)$. The distribution $p(\theta/x, y)$ is the posterior distribution of the parameters, conditional on the observed data and the latent variables. Similarly, the distribution $p(x/\theta, y)$ is the smoothing distribution of the latent variables given the parameters. The MCMC approach therefore extends model findings relative to non-linear optimizers by breaking “curse of dimensionality” by transforming a higher dimensional problem, sampling from $p(\theta_1, \theta_2)$, into easier problems, sampling from $p(\theta_1/\theta_2)$ and $p(\theta_2/\theta_1)$ (using the Besag (1974) formula).

The successful use of this version of the SV model for the carbon markets, suggests positive serial correlation in the mean, volatility tends to cluster and negative correlation between the mean and volatility induces asymmetry (the leverage effect). Although price processes are hardly predictable, the variance of the forecast error is time dependent and can be estimated by means of observed past variations. The result suggests that volatility can be forecasted. Moreover, observed volatility clustering induce an unconditional distribution of returns at odds with the hypothesis of normally distributed price changes. The stochastic volatility models are therefore an area in empirical financial data modelling that is fruitful as a practical descriptive and forecasting device for all participants/managers in the financial services sector together with a special emphasis on

risk management (forecasting/ re-projections and VaR/Expected shortfall) and portfolio management (option contract prices and Greek letters).

Irrespective of markets and contracts, Monte Carlo Simulations should lead us to more insight of the nature of the price processes describable from stochastic volatility models. Our findings suggest that the European carbon markets are well described using two-factor scientific SV models. The Bayesian MCMC methodology, the parallel CPU processing and the M-H algorithm are computationally intensive for estimation, inference and assessment of model adequacy. Based on the simulated series the methodology extends information to the market participants (density forecasts); the conditional mean and volatility (conditional moments), forecasting conditional volatility (filtering), conditional variance functions for asymmetry (smoothing) and multiple-ahead dynamics (mean reversion/persistence analysis).

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NOTES

¹ The Intercontinental Commodity Exchange report 91,5% of the total European Carbon trading volume (2013).

² NASDAQ OMX: www.nasdaq.com; European Energy Exchange: www.eeg.de; Intercontinental Exchange: www.theice.com.

³ *Source: Point Carbon*. In 2010, trading volumes across the entire carbon market transacted 7 billion tonnes of carbon dioxide equivalent (CO₂e) with a value of € 92 billion. Total transaction volume in the global carbon market went down 12% in 2009. However, the total value of the market in 2010 went up by 1% year-on-year.

⁴ The co-integration argument is analysed in several international studies. The most comprehensive studies are Westgaard et al. (2010), Jong and Schneider (2009), Veka et al. (2011) and references therein.

⁵ The methodology is designed for estimation and inference for models where (1) the likelihood is not available, (2) some variables are latent (unobservable), (3) the variables can be simulated and (4) there exist a well-specified and adequate statistical model for the simulations. The methodologies (General Scientific Models (GSM) and Efficient Method of Moments (EMM)) are general-purpose implementation of the Chernozhukov and Hong estimator (2003). That is, the applications for methodologies are not restricted to simulation estimators.

⁶ Using only market traded products avoid the continuous-time no-arbitrage condition.

⁷ See for example Benz & Trück (2009) and Paoletta and Taschini (2006) for applications in energy markets. ARCH studies of energy markets are numerous and are often described as SV, but we do not follow that nomenclature. In the SV approach the predictive distribution of returns is specified indirectly, via the structure of the model, rather than directly as for ARCH. However, the accompanying statistical model for the MCMC estimation in this paper is a nonparametric-ARCH model.

⁸ High frequency data based on the concept realized volatility is tied to continuous-time processes and SV models.

⁹ See www.econ.duke.edu/webfiles/arg for software and applications of the MCMC Bayesian methodology. All models are coded in C++ and executable in both serial and parallel versions (OpenMPI).

¹⁰ The original statistical fit-model and the re-tuned assessment model with $\kappa = 20$, give roughly the same parameter (η) results.

¹¹ A quadratic fit is used and solved for the point where the quadratic equals the chi-square 1 df. critical value 3.841.

¹² To avoid too dense plots only every 1000 (steps=1000) datum are plotted (250k/1000 =250 plots).

¹³ When up-to-date re-projected volatility is calculated and available the pricing can easily be implemented in for example the Excel spreadsheet for any derivative function.

Table 1. Characteristics of the Carbon Future Market

| Mean / Mode | Median Std.dev. | Maximum/ Minimum | Moment Kurt/Skew | Quantile Kurt/Skew | Quantile Normal | Anderson Darling | Serial dependence | |
|-----------------------------------|--------------------|---------------------|---------------------|-----------------------|--------------------|---------------------|-------------------|--------------------------------|
| -0.12863 | 0.00000 | 22.8378 | 6.58824 | 0.27955 | 4.2953 | 4.59075 | Q(12) 23.9950 | Q ² (12) 365.857 |
| 0.00000 | 2.98187 | -16.6157 | 0.11760 | -0.01605 | {0.1168} | {0.0000} | {0.0204} | {0.0000} |
| BDS-statistic ($\varepsilon=1$) | | | KPSS (Stationary) | | | Augmented | ARCH | VaR 2.5% / |
| m=2 | m=3 | m=4 | m=5 | Level | Trend | DF-test | (12) | CVaR 2.5% |
| 16.6788 | 23.5820 | 30.1427 | 38.4401 | 0.14330 | 0.14340 | -56.0675 | 594.675 | -6.704 |
| {0.0000} | {0.0000} | {0.0000} | {0.0000} | {0.4121} | {0.0568} | {0.0000} | {0.0000} | -8.984 |

The numbers in braces denote p -values for statistical significance.

Table 2. Carbon characteristics of the Statistical Semi-parametric Model Residuals

| Mean / Mode | Median / Std.dev. | Maximum/ Minimum | Moment Kurt/Skew | Quantile Kurt/Skew | Quantile Normal | Anderson Darling | Serial dependence Q(12) | Q ² (12) |
|-----------------------------------|----------------------|---------------------|---------------------|-----------------------|--------------------|---------------------|----------------------------|---------------------|
| 0.00540 | 0.04768 | 4.34449 | 2.53610 | 0.07015 | 0.56074 | 0.6158701 | 22.8050 | 7.7449 |
| 0.43520 | 1.00066 | -5.63364 | -0.31283 | -0.03709 | {0.7555} | {1.1366} | {0.0423} | {0.8047} |
| BDS-statistic ($\varepsilon=1$) | | | | ARCH | RESET | Joint | VaR | CVaR |
| m=2 | m=3 | m=4 | m=5 | (12) | (12;6) | Bias | 2.5%/0.5% | 2.5%/0.5% |
| 0.056986 | 0.192762 | -0.072207 | 0.032576 | 0.5461 | 7.46051 | 2.909234 | -2.000087 | -2.779752 |
| {0.3983} | {0.3916} | {0.3979} | {0.3987} | {0.4599} | {0.8257} | {0.4058} | -3.219097 | -4.207360 |

The numbers in braces denote p -values for statistical significance.

Table 3. Scientific Stochastic Volatility Characteristics: Carbon θ -parameters*

| Carbon Front December General Scientific Model. Parallell. | | | | Statistical Model SNP-1111A000 - fit model | | |
|--|-------------------|-------------|---------------|--|-----------------------|----------------|
| Parameter values | Scientific Model. | Standard | error | Parameters | Semiparametric-GARCH. | |
| θ | Mode | Mean | | η | Mode | Standard error |
| θ_1, a_0 | -0.0344470 | -0.0311740 | 0.0428120 | η_1 a0[1] | 0.0071800 | 0.0142900 |
| θ_2, a_1 | 0.0326050 | 0.0324780 | 0.0195720 | η_2 a0[2] | -0.0315000 | 0.0296300 |
| θ_3, b_0 | 0.7923400 | 0.7812700 | 0.0976260 | η_3 a0[3] | -0.0353700 | 0.0148600 |
| θ_4, b_1 | 0.9847500 | 0.9815600 | 0.0095960 | η_4 a0[4] | 0.0864600 | 0.0179400 |
| θ_5, s_1 | 0.0755390 | 0.0754390 | 0.0131280 | | | |
| θ_6, s_2 | 0.2323200 | 0.2258900 | 0.0529770 | η_6 B(1,1) | 0.0338300 | 0.0296000 |
| θ_7, r_1 | -0.5196300 | -0.5070600 | 0.1217300 | η_7 R0[1] | 0.0932200 | 0.0170800 |
| | | | | η_8 P(1,1) | 0.2149000 | 0.0492200 |
| log sci_mod_prior | | 0.0327854 | | η_9 Q(1,1) | 0.9549700 | 0.0089300 |
| log stat_mod_prior | | 0 | $\chi^2(2) =$ | η_{10} V(1,1) | -0.2884900 | 0.0497200 |
| log stat_mod_likelihood | | -1397.78670 | -3.4027 | | | |
| log sci_mod_posterior | | -1397.75392 | {0.182437} | | | |

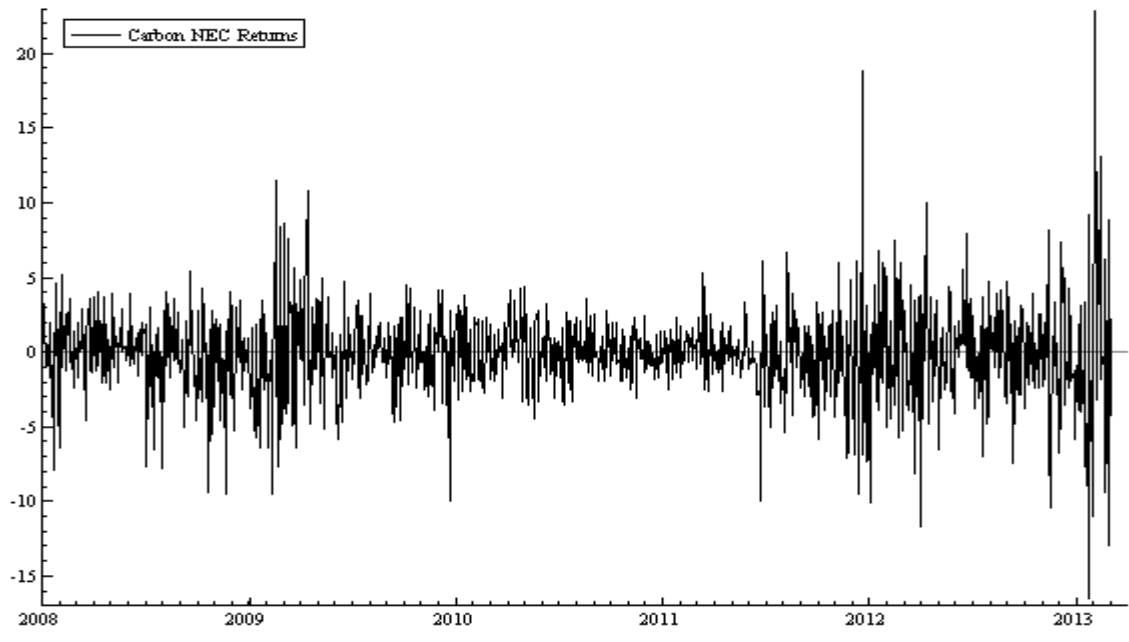
*The parameter c_0 from the original model (page10) is fixed equal to zero (0). The constant part of volatility is therefore all in the b_0 parameter. The parameters r_2 and r_3 are initially free parameters but from the estimation they are very close to zero (0). In the final parallel estimation the r_2 and r_3 parameters are both fixed and equal to zero (0). The $\chi^2(2)$ reports a satisfactory fit of the optimal 7 parameter SV model. The number in braces denotes p -values for statistical significance.

Table 4. Carbon Confidence intervals using criterion differences

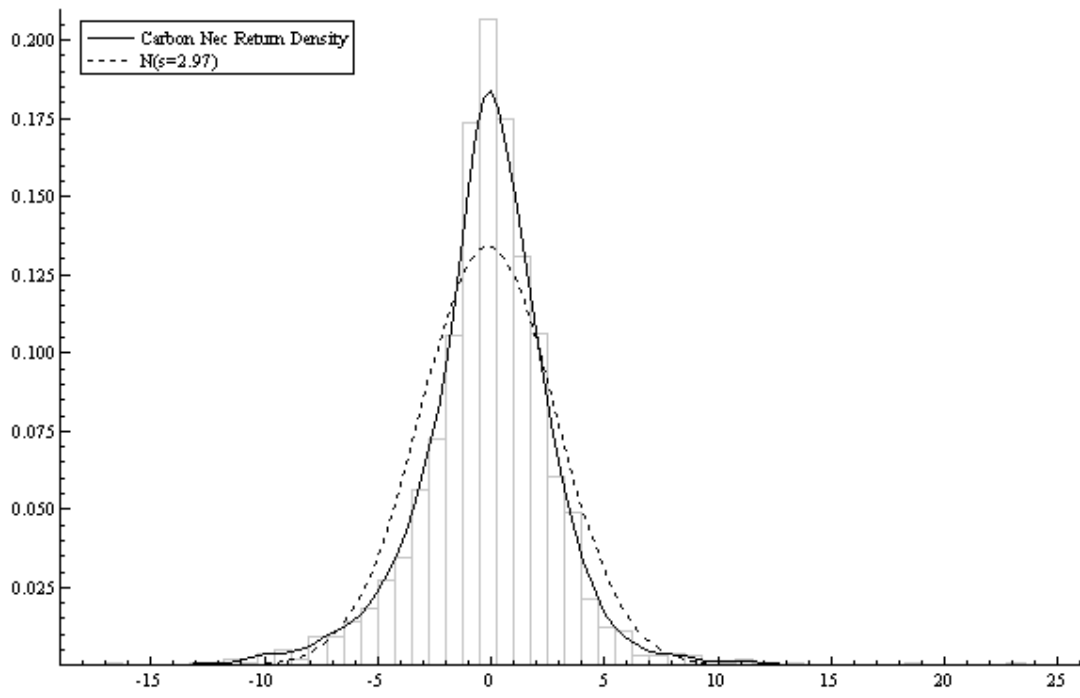
| SV Model | | 95% Lower | 95% Upper | SV Model | | 95% Lower | 95% Upper |
|--------------|------------|----------------|----------------|--------------|------------|---------------|----------------|
| Coefficients | Optimum | Critical Point | Critical Point | Coefficients | Optimum | CriticalPoint | Critical Point |
| θ_1 | -0.0344470 | -0.0857 | 0.02126 | θ_5 | 0.0755390 | 0.04984 | 0.11435 |
| θ_2 | 0.0326050 | -0.01938 | 0.06483 | θ_6 | 0.2323200 | 0.16315 | 0.35343 |
| θ_3 | 0.7923400 | 0.64539 | 0.91252 | θ_7 | -0.5196300 | -0.73486 | -0.40349 |
| θ_4 | 0.9847500 | 0.87345 | 0.99545 | | | | |



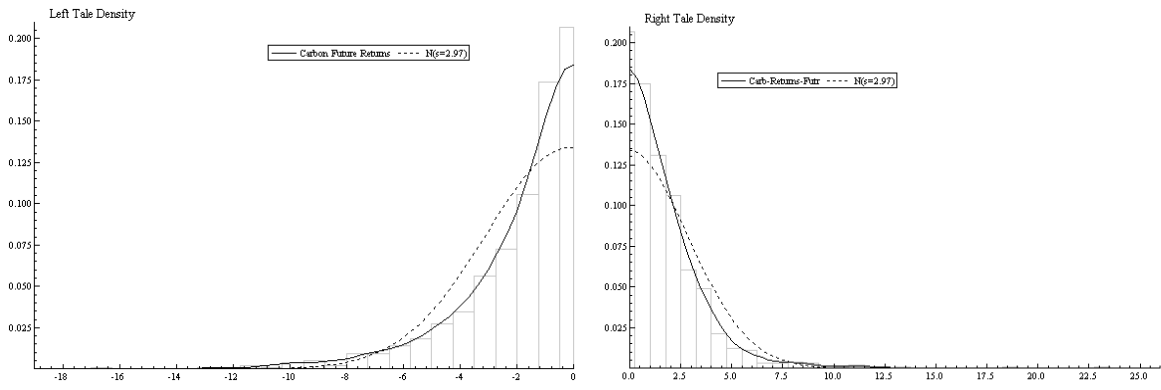
a. Carbon Contract Prices Level Plot 2008-2013



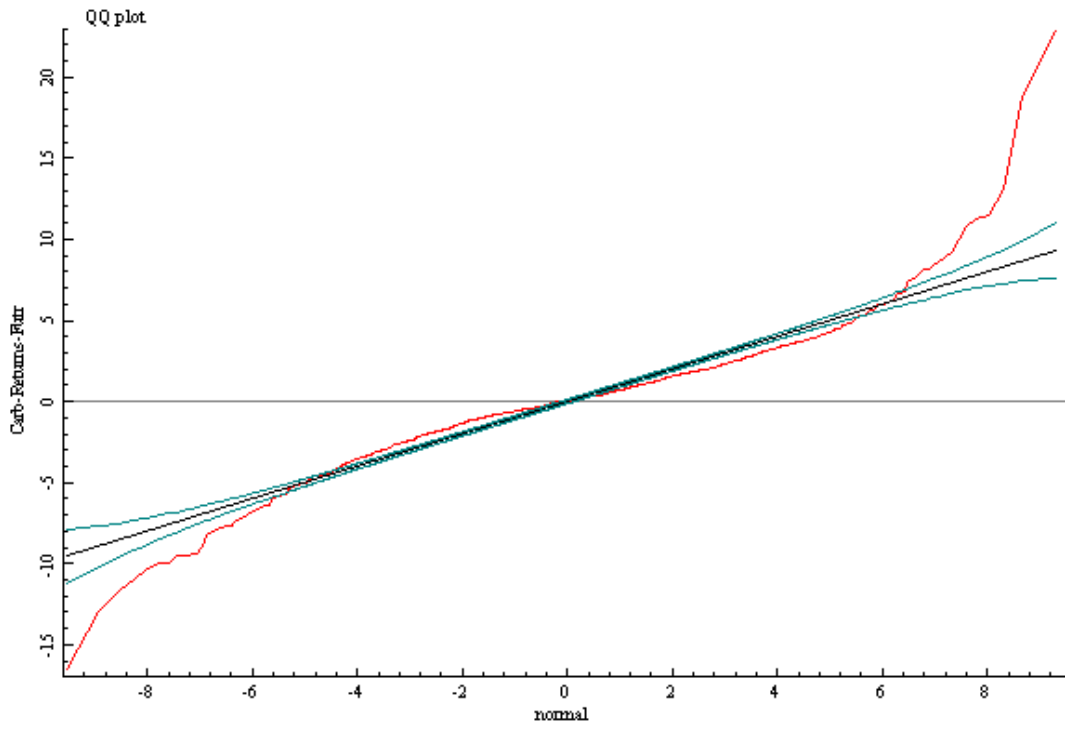
b. Carbon Contracts Returns Plot 2008-2013



c. Carbon Contracts Returns Distribution Plot

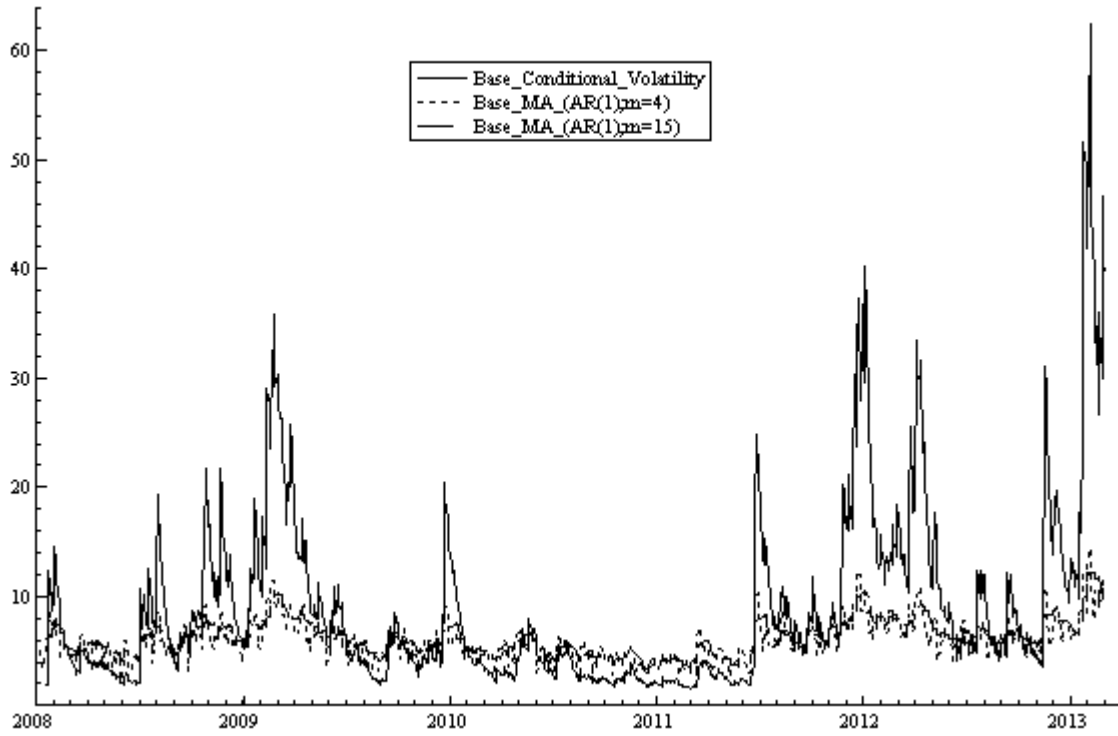


d. Carbon Contracts Returns Tails-distribution plots

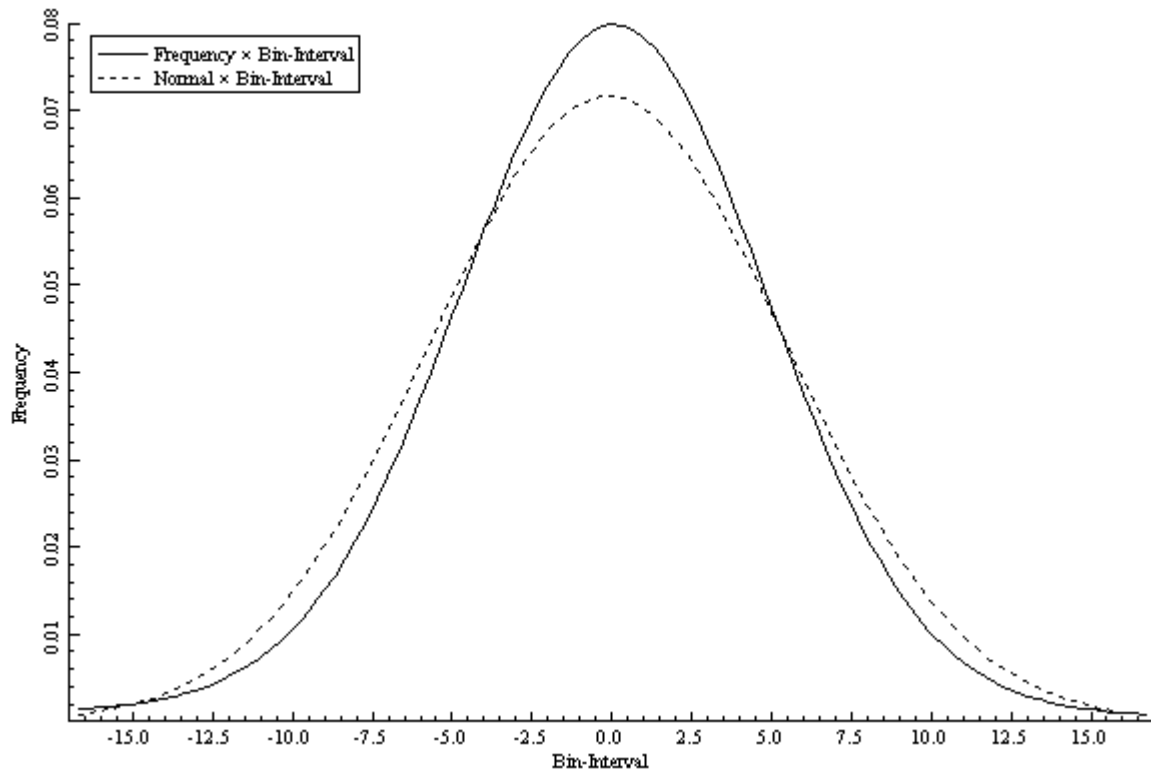


e. Carbon Contracts Returns QQ -plot

Figure 1. Carbon Contract Characteristics for the period 2008-2013

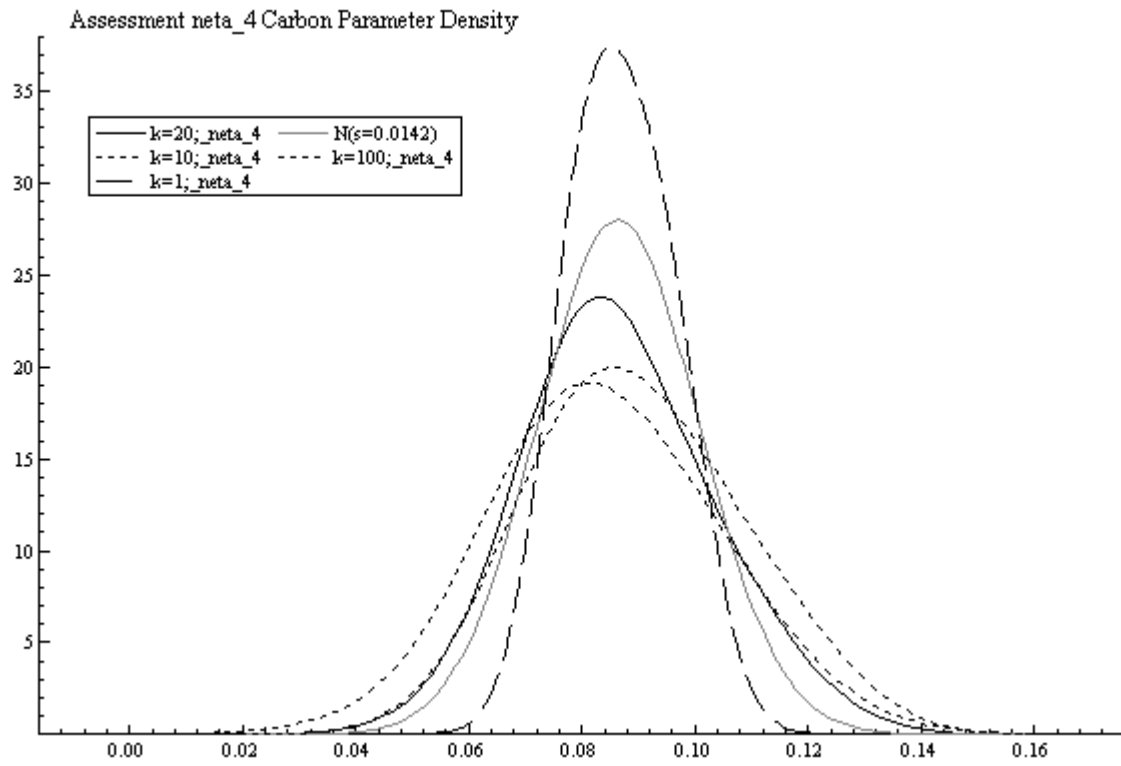


a. Carbon Projected Conditional Volatility & Moving Average ($m=4$ and 15)

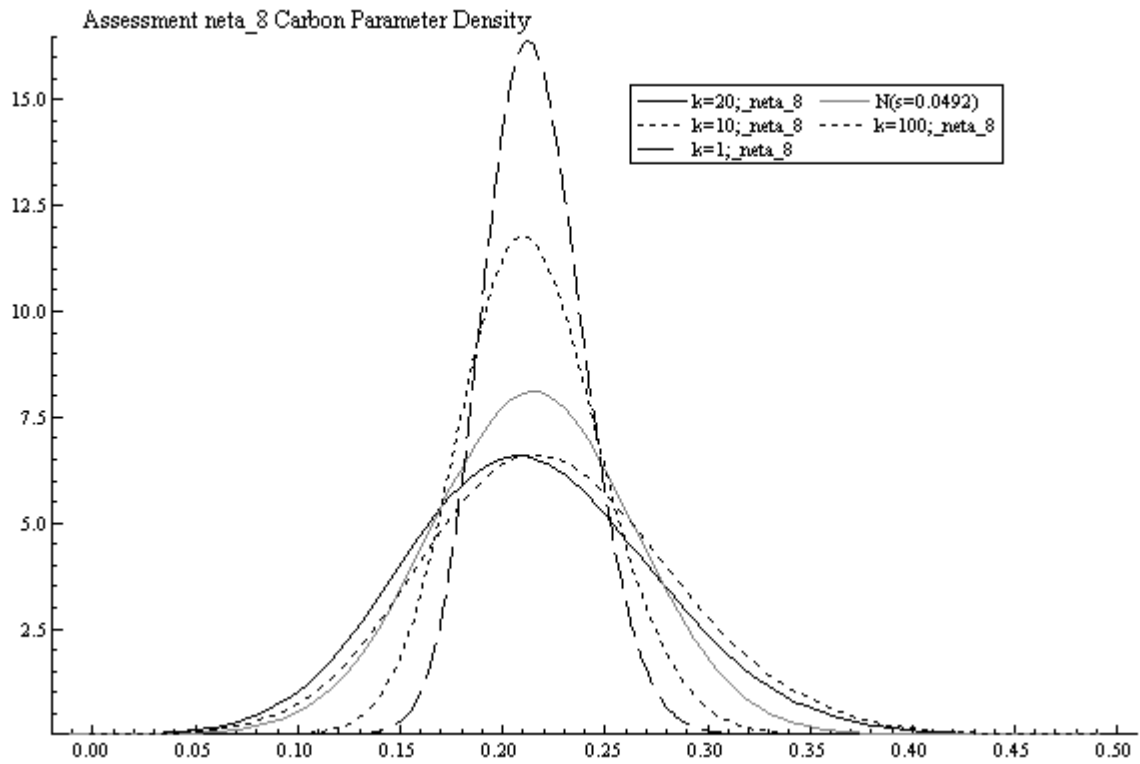


b. One-step-ahead Carbon Density

Figure 2. Carbon contract characteristics of the Statistical GSM Score

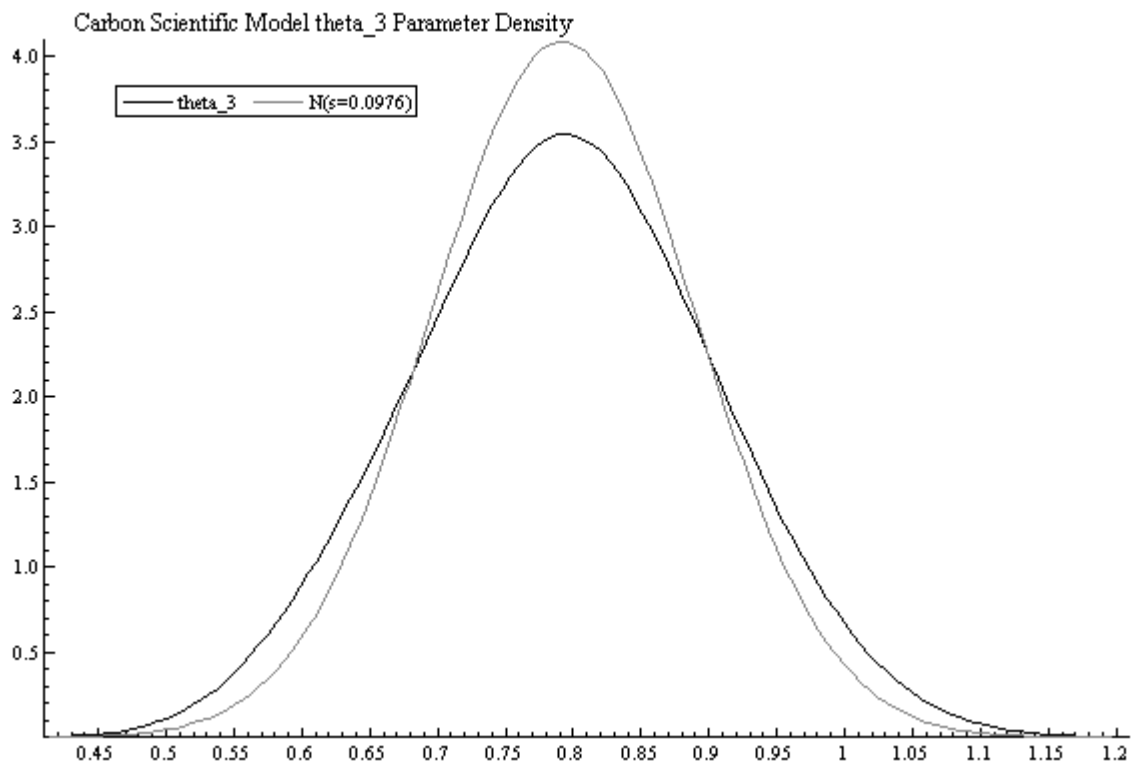


a. η_4 parameter model assessment

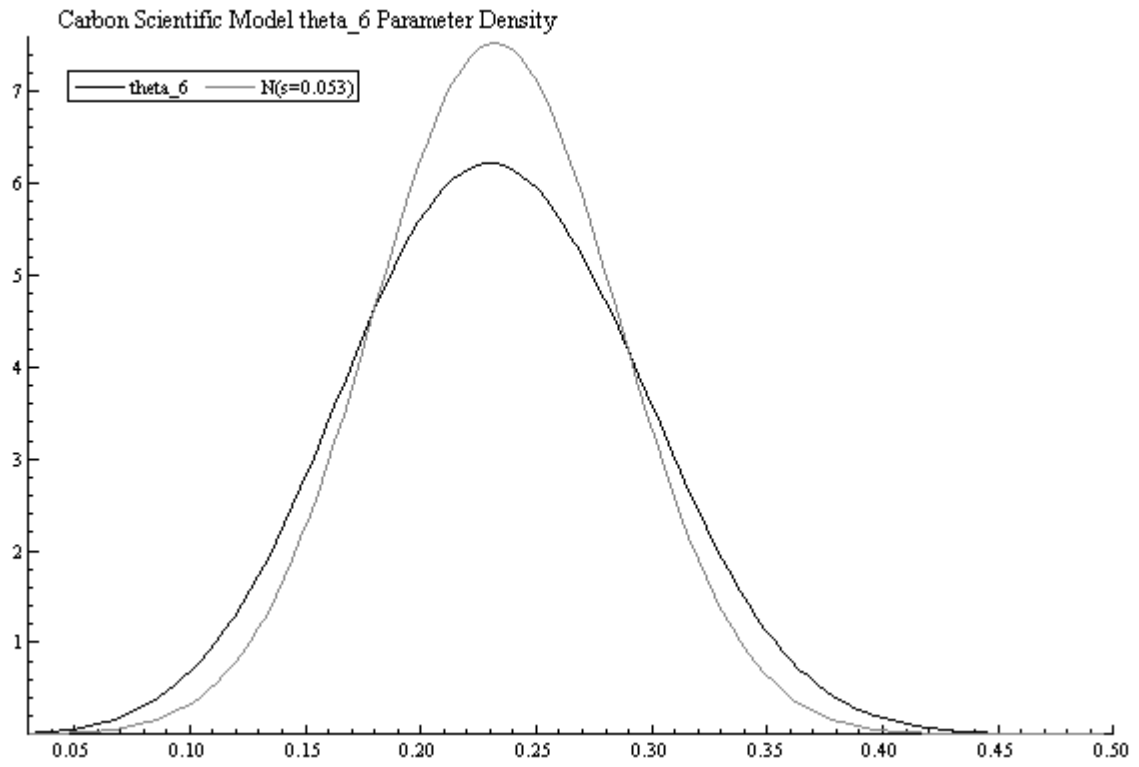


b. η_{s8} -parameter model assessment

Figure 3. Carbon contract η - parameter distributions for model assessment: $\kappa = 1, 10, 20$ and 100 . Every 25th observation is used from a sample of 250000 (10000 observations are used for each plot).



a. Carbon SV model parameter: θ_3



b. Carbon SV model parameter: θ_6

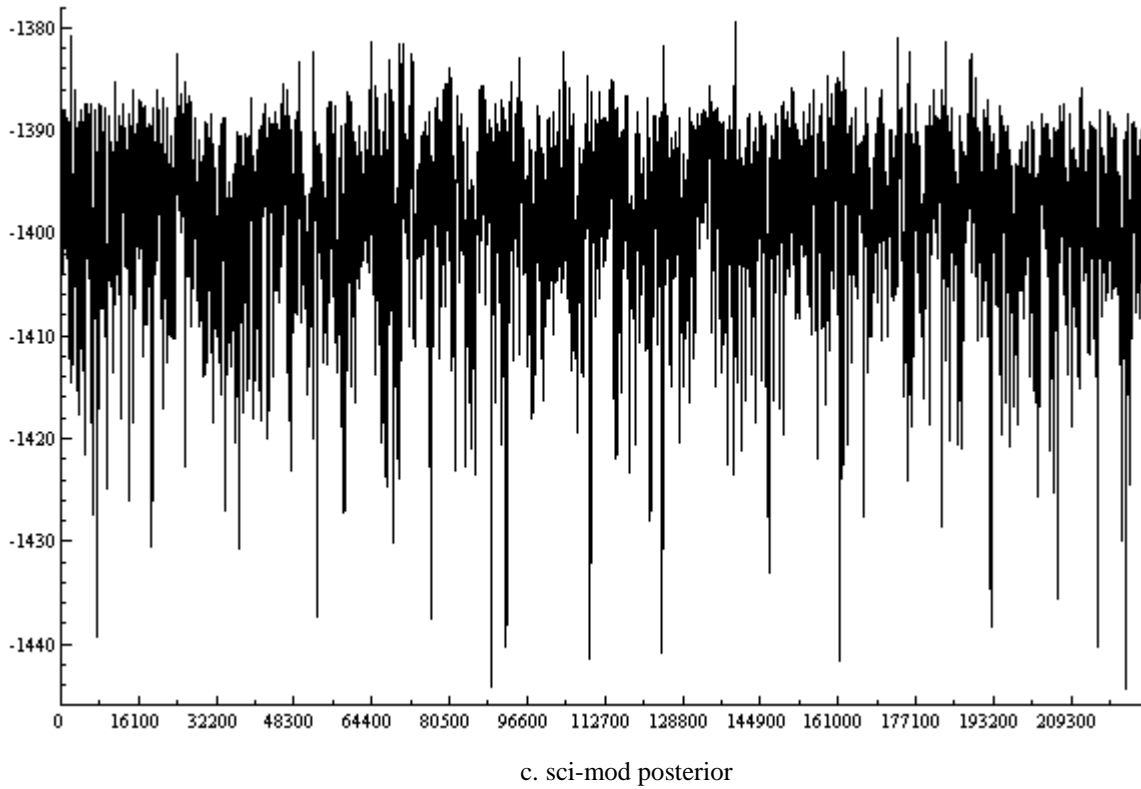
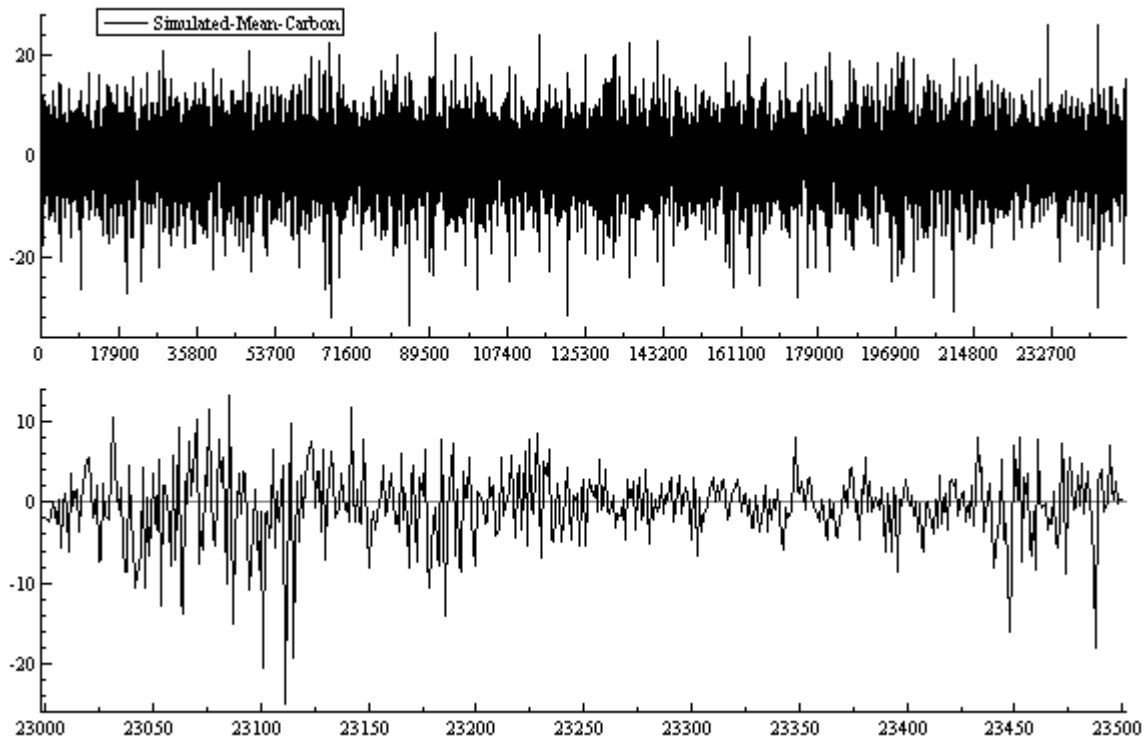
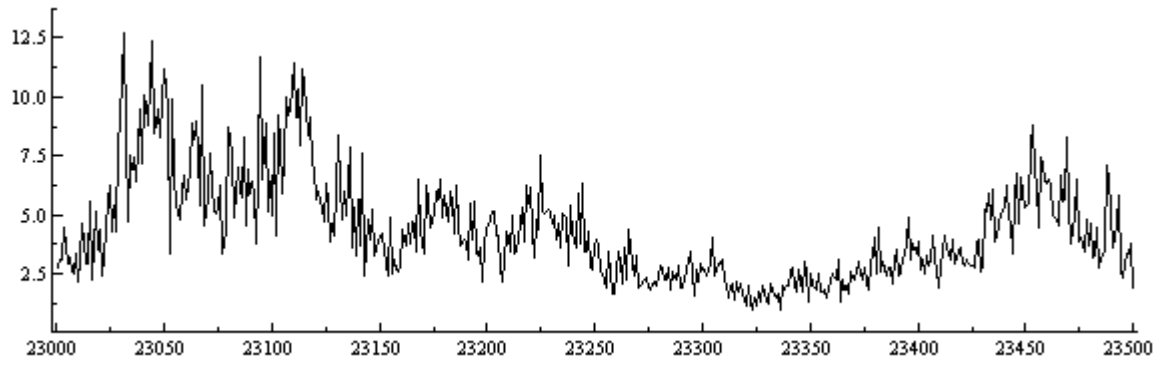
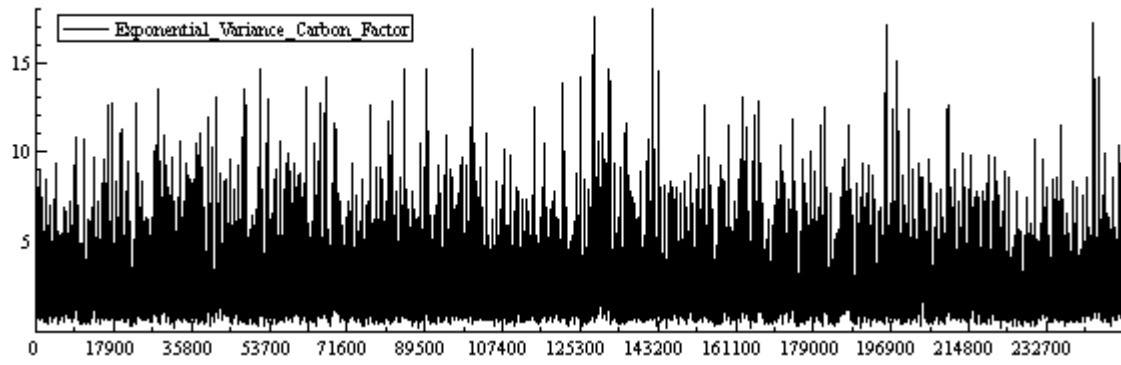


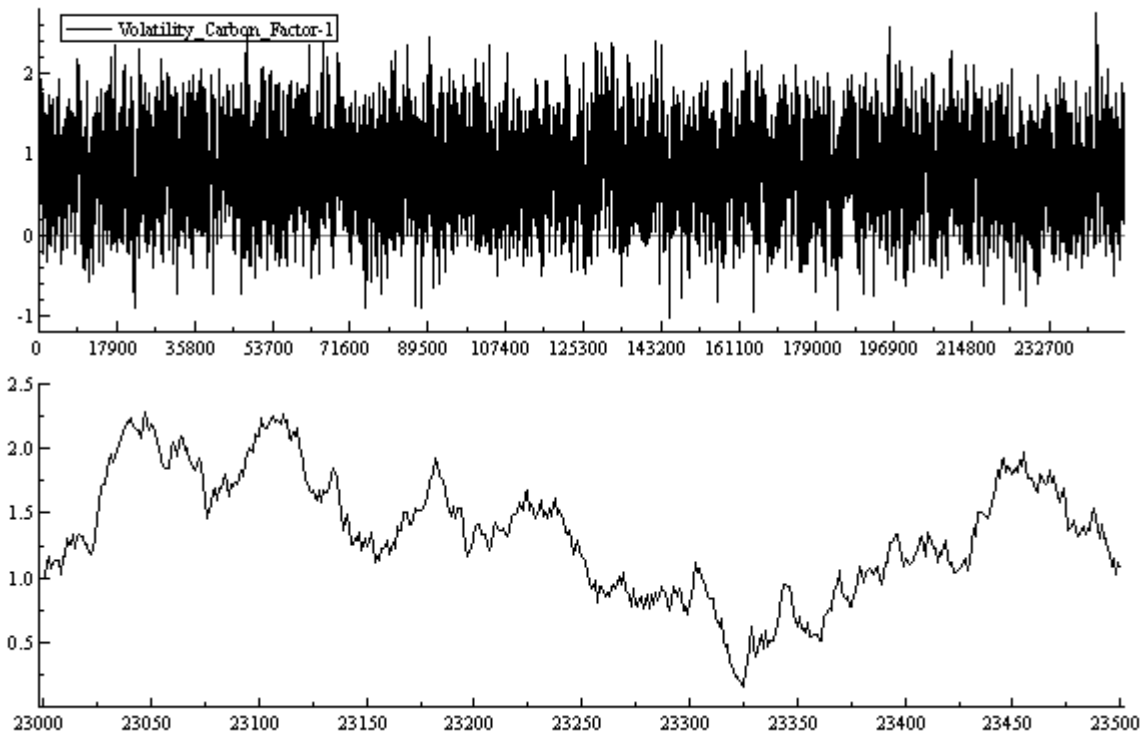
Figure 4. Carbon contract θ - parameter paths and distributions for a 25 CPU parallel-run. Every 25th observation is used implying a total sample of 250 k (10 k observations are used in each plot).



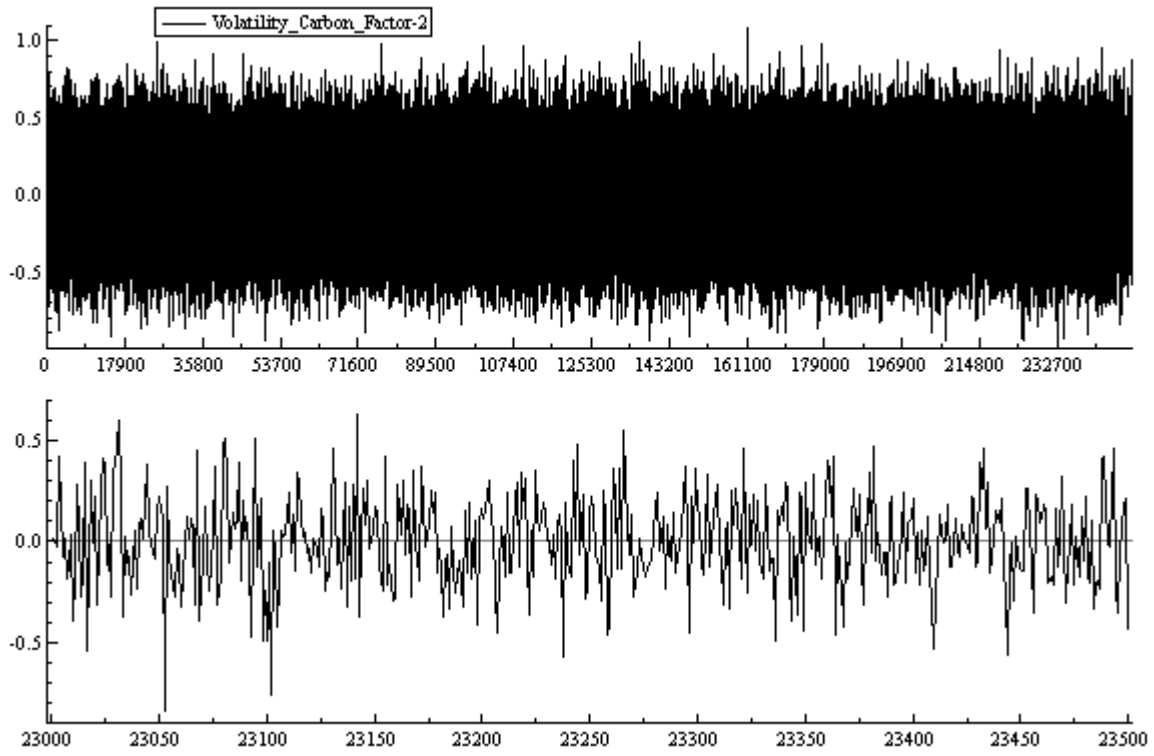
a. Carbon simulated return series (250 k and 0.5 k)



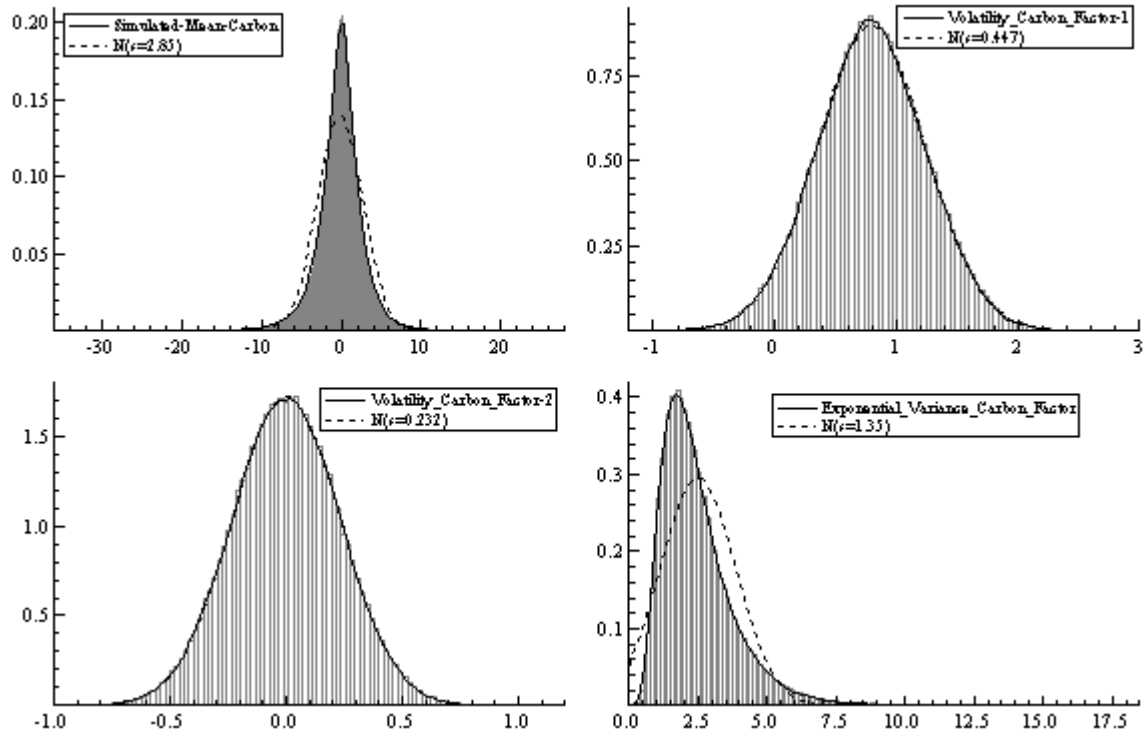
b. Carbon simulated Exp(Volatility) series (250 k and 0.5k)



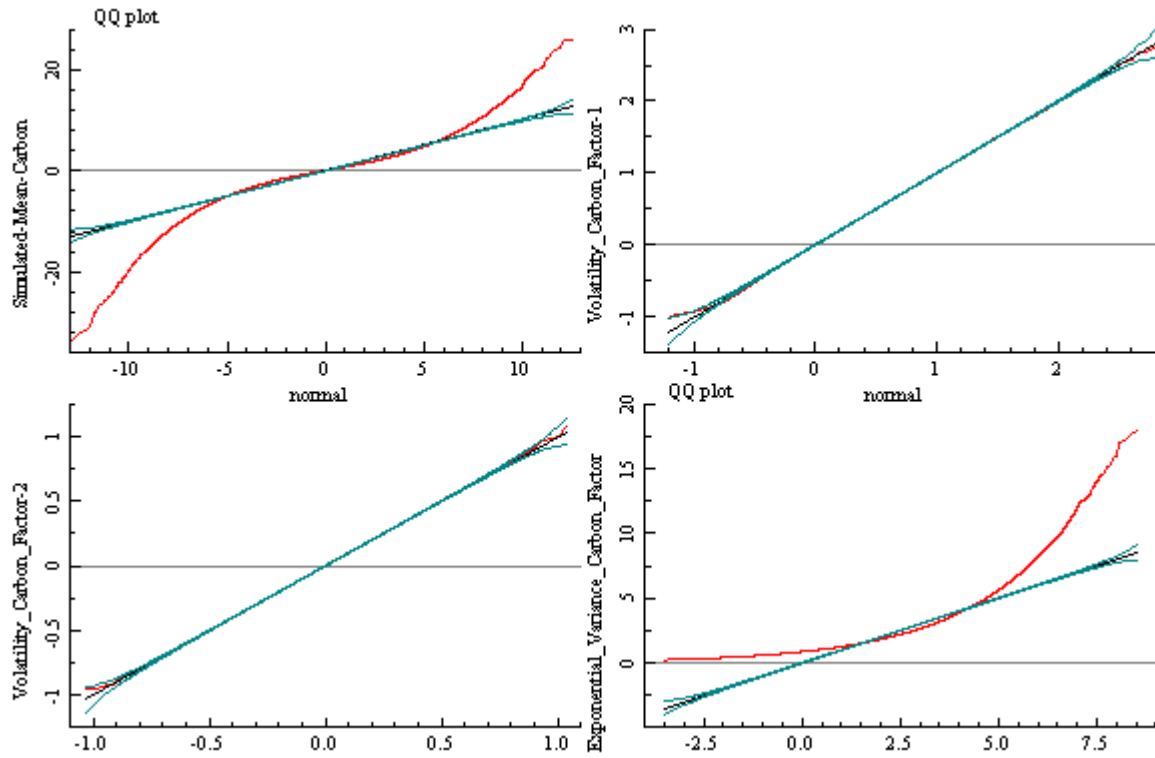
c. Carbon simulated Volatility Factor 1 series (250 k and 0.5 k)



d. Carbon simulated Volatility Factor 2 series (250 k and 0.5 k)

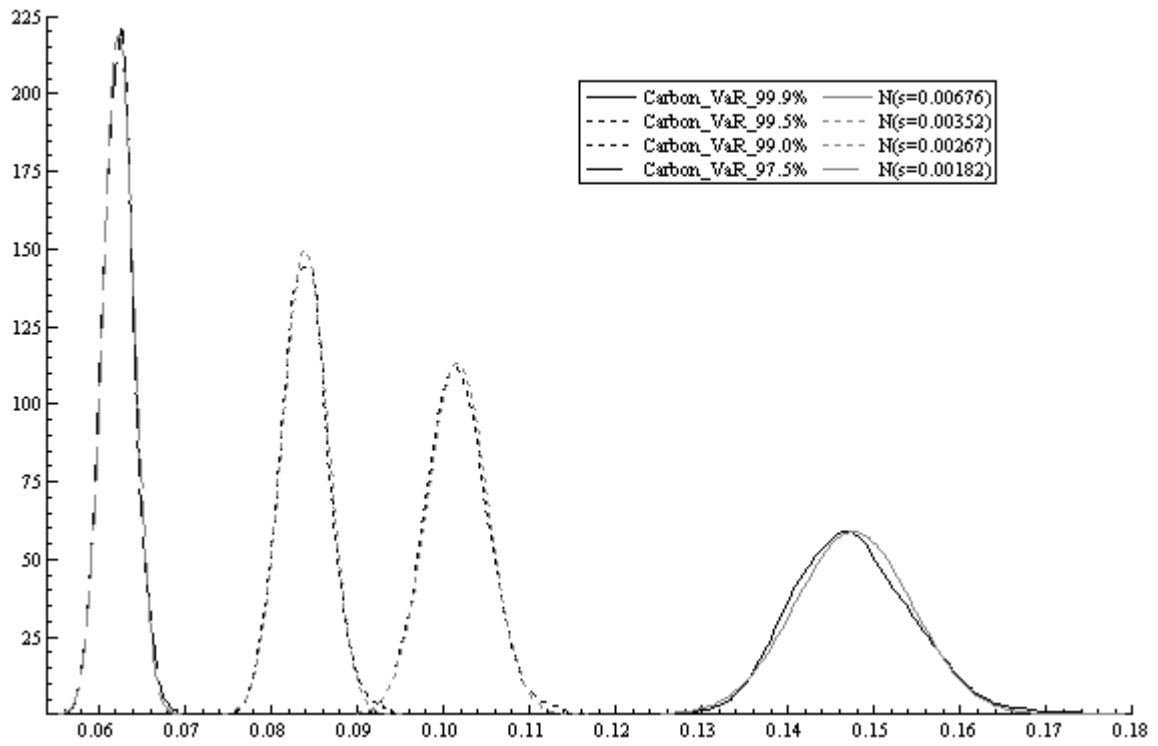


e. Carbon distribution Mean and Volatility

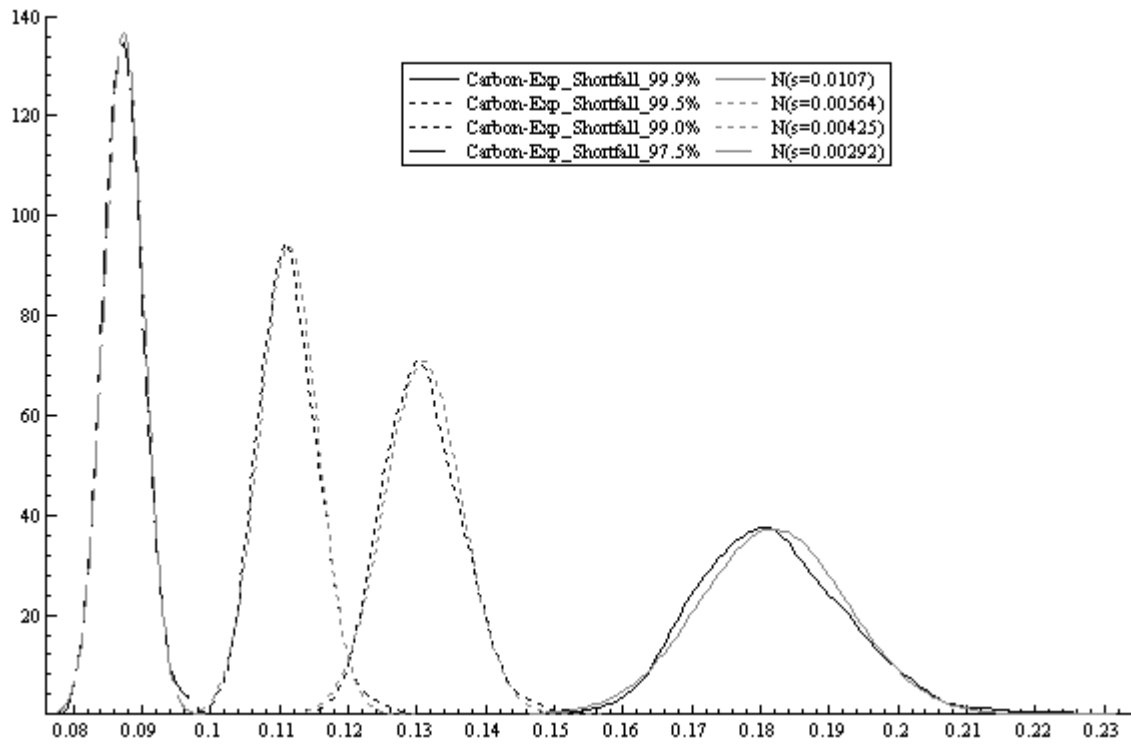


f. *QQ*-plots for Carbon Mean and Volatility

Figure 5. Carbon contract SV model characteristics: Full and sub sample mean (a) and volatility (b). Full and sub sample volatility factors(c and d). Mean and volatility densities (e) and *QQ* plots (f)

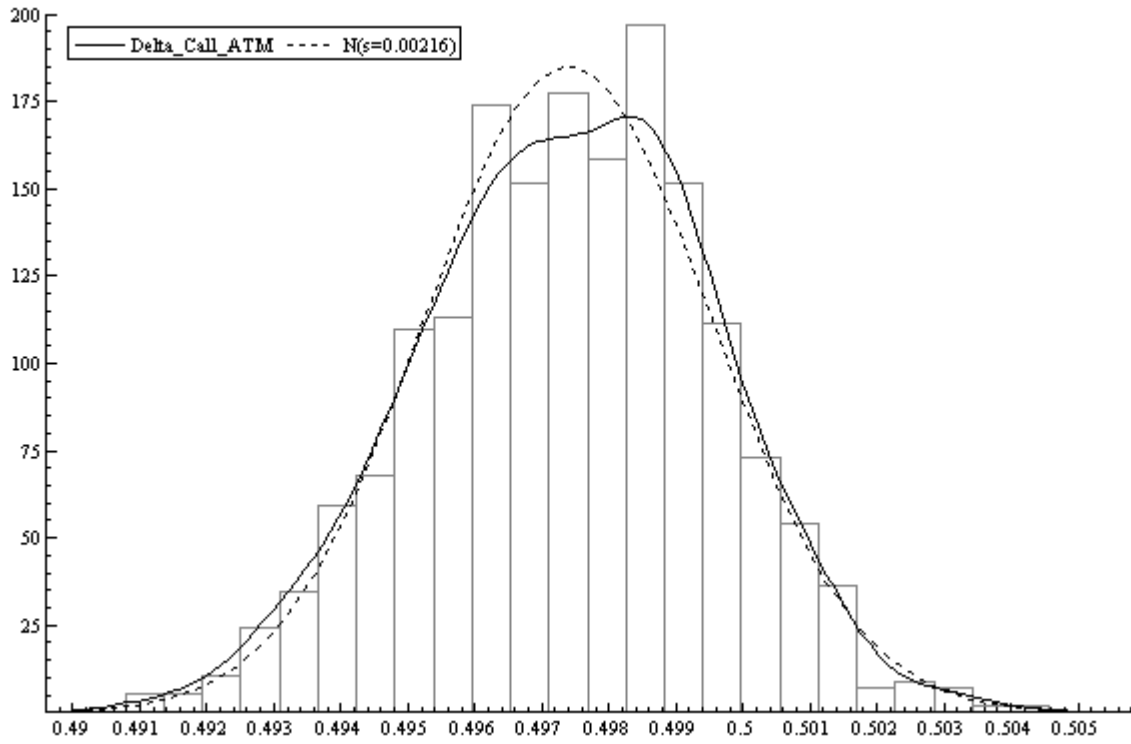


a. Value-at-risk (VaR) for 99.9%, 99.5%, 99% and 97.5% densities

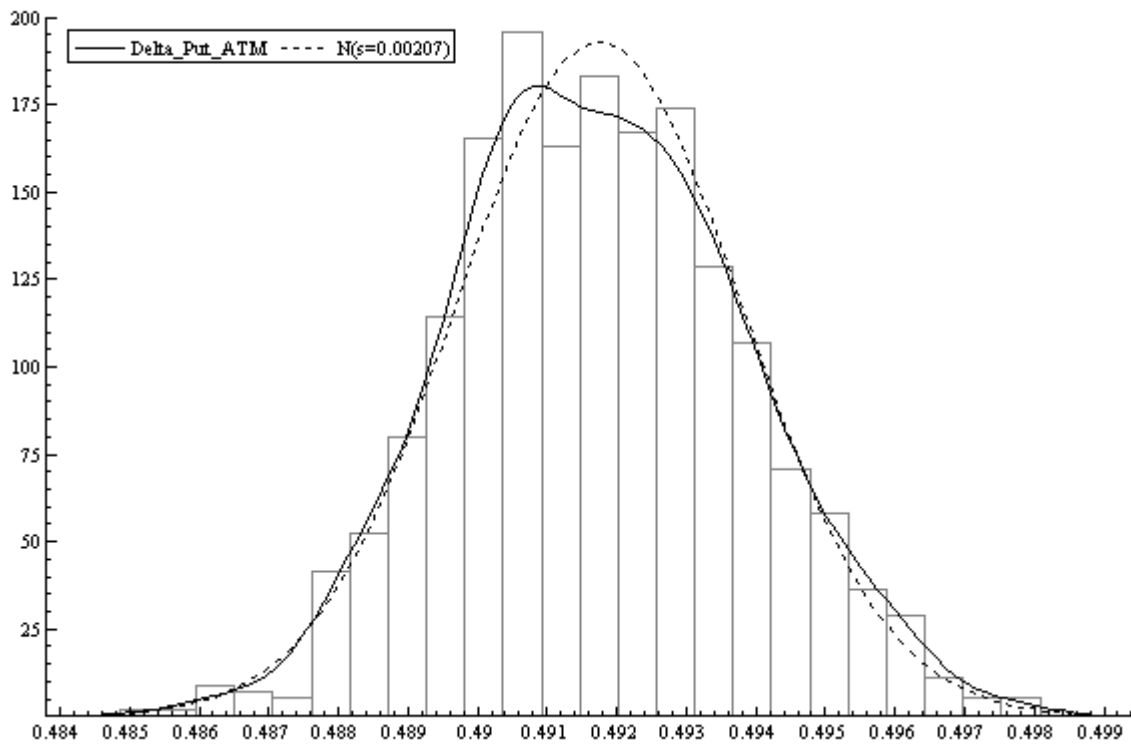


b. Conditional value-at-risk (CVaR) for 99.9%, 99.5%, 99% and 97.5% densities

Figure 6 Carbon Contracts 10 *k* iterations: *VaR* (a) and Expected Shortfall (CVaR) (b)

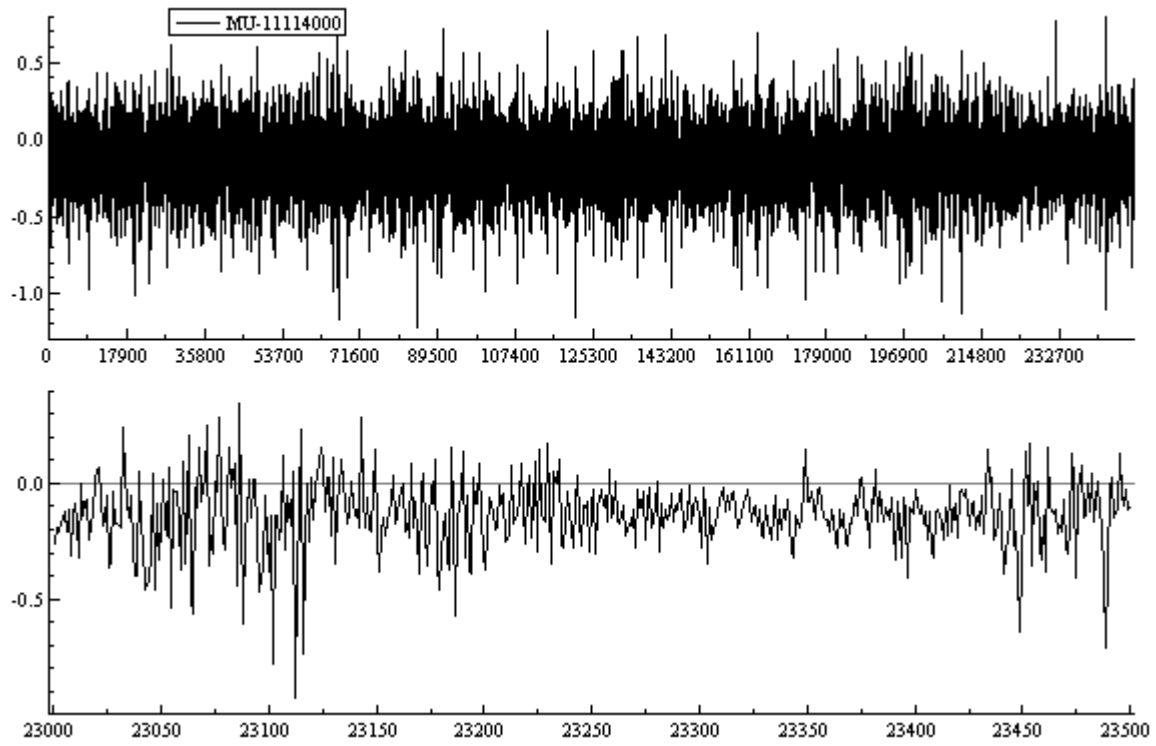


a. Delta for an ATM Call Carbon option

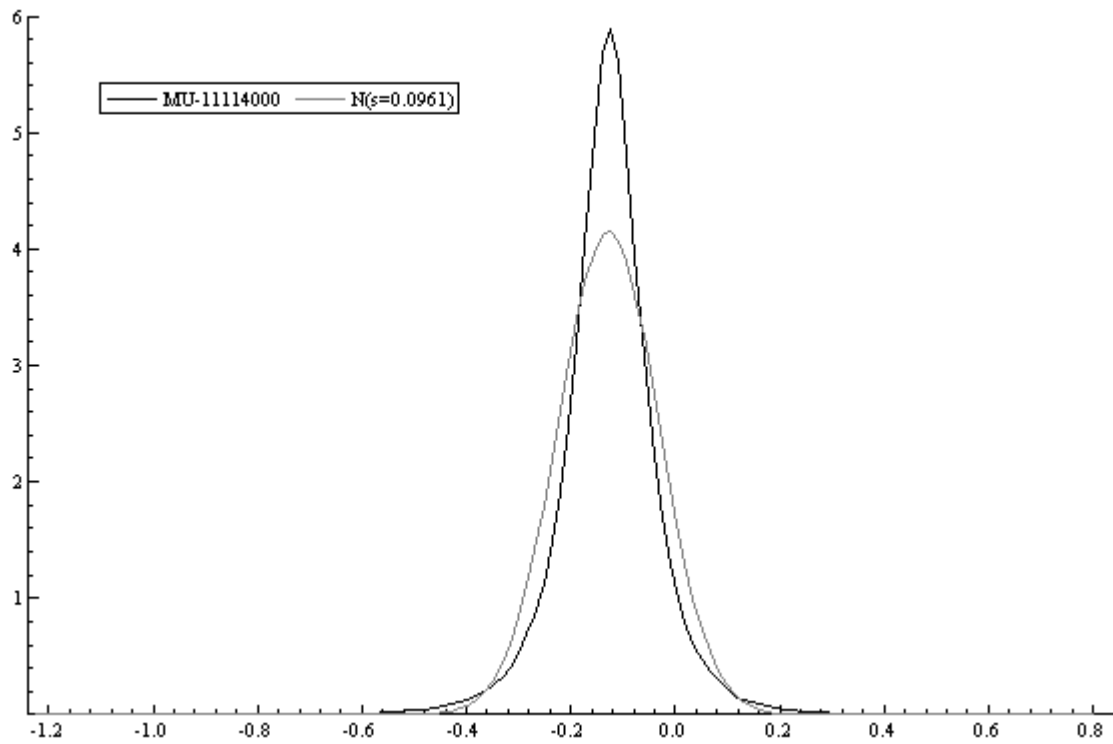


b. Delta for an ATM Carbon Put option

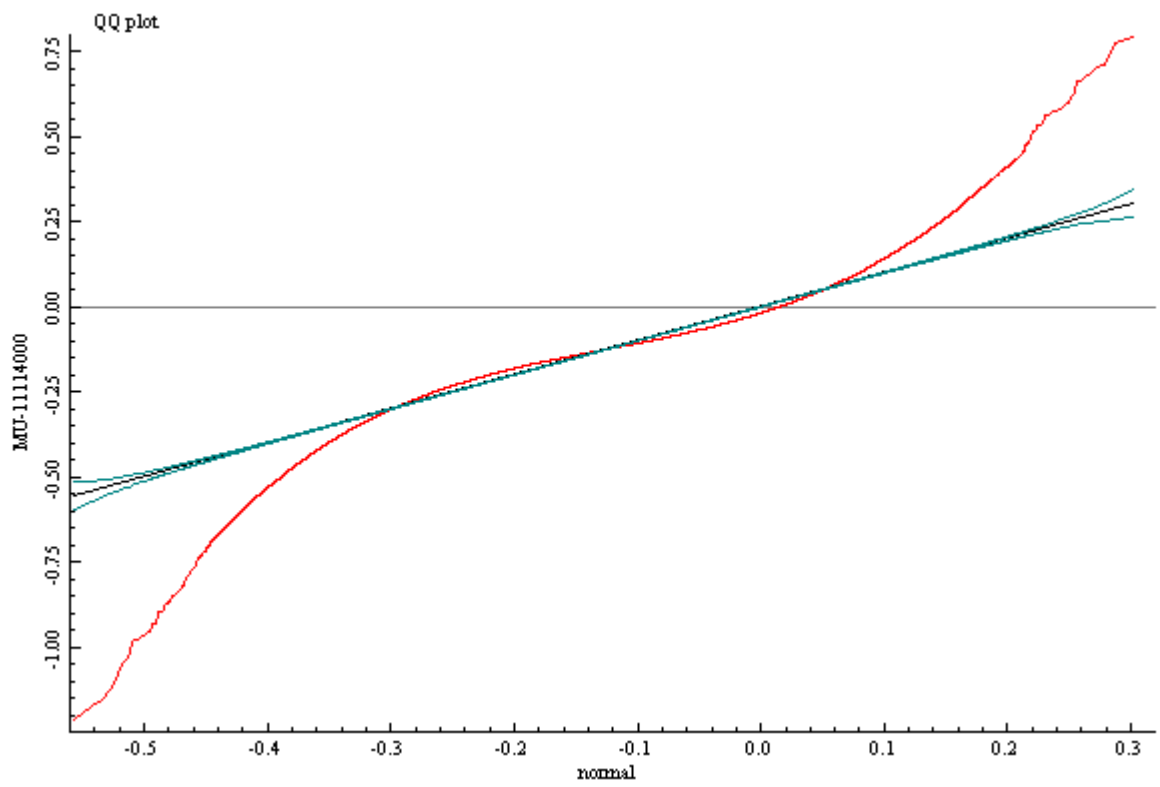
Figure 7. Carbon Contracts: Delta densities for Call (a) and Put (b) ATM options. 10 *k* iterated forecasts



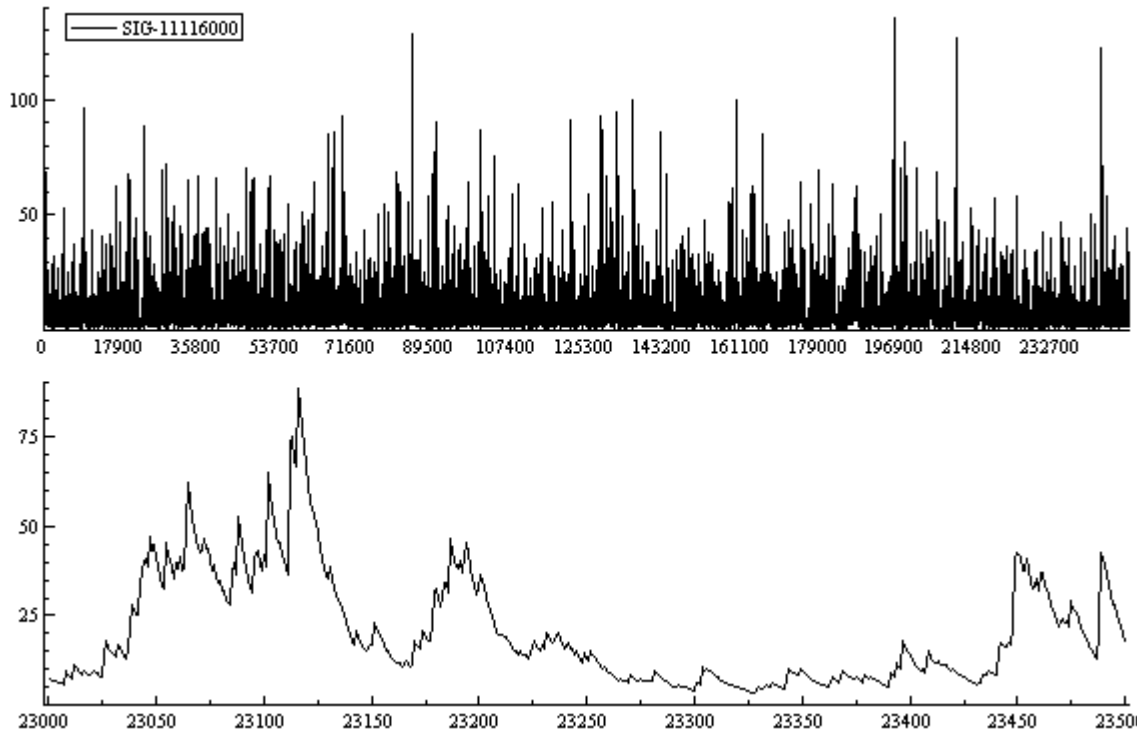
a. One-step-ahead Carbon Conditional Mean (250 k and 0.5 k)



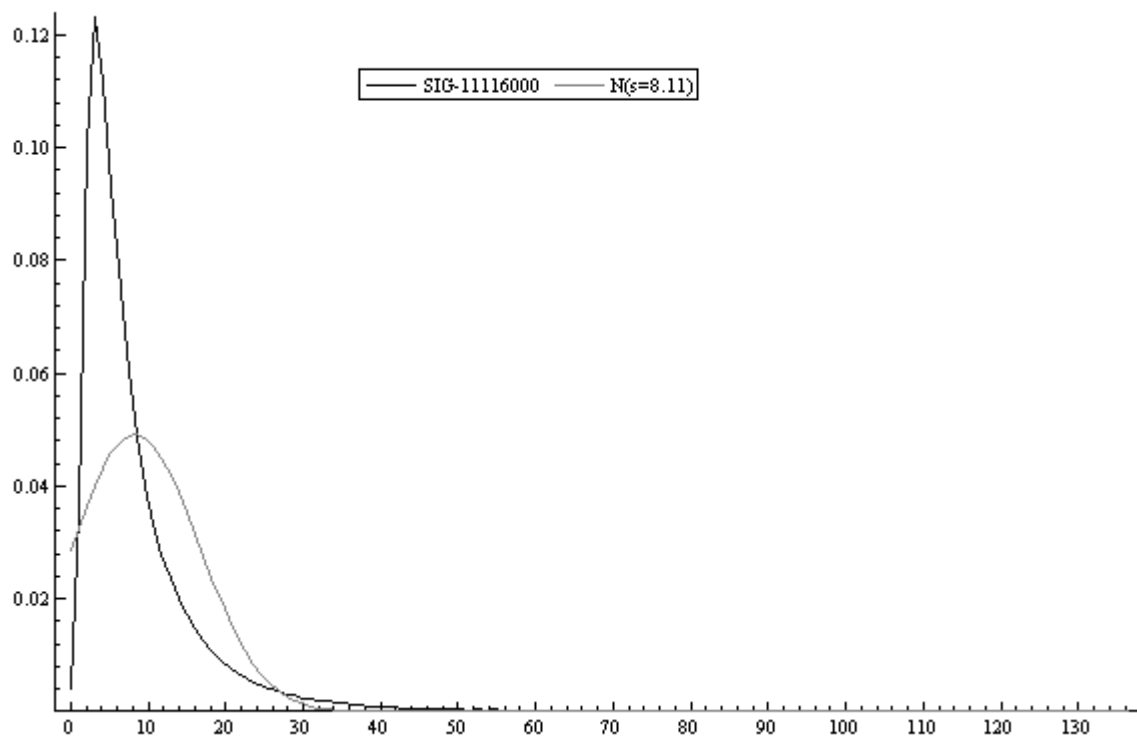
b. One-step-ahead Carbon Conditional Mean Density



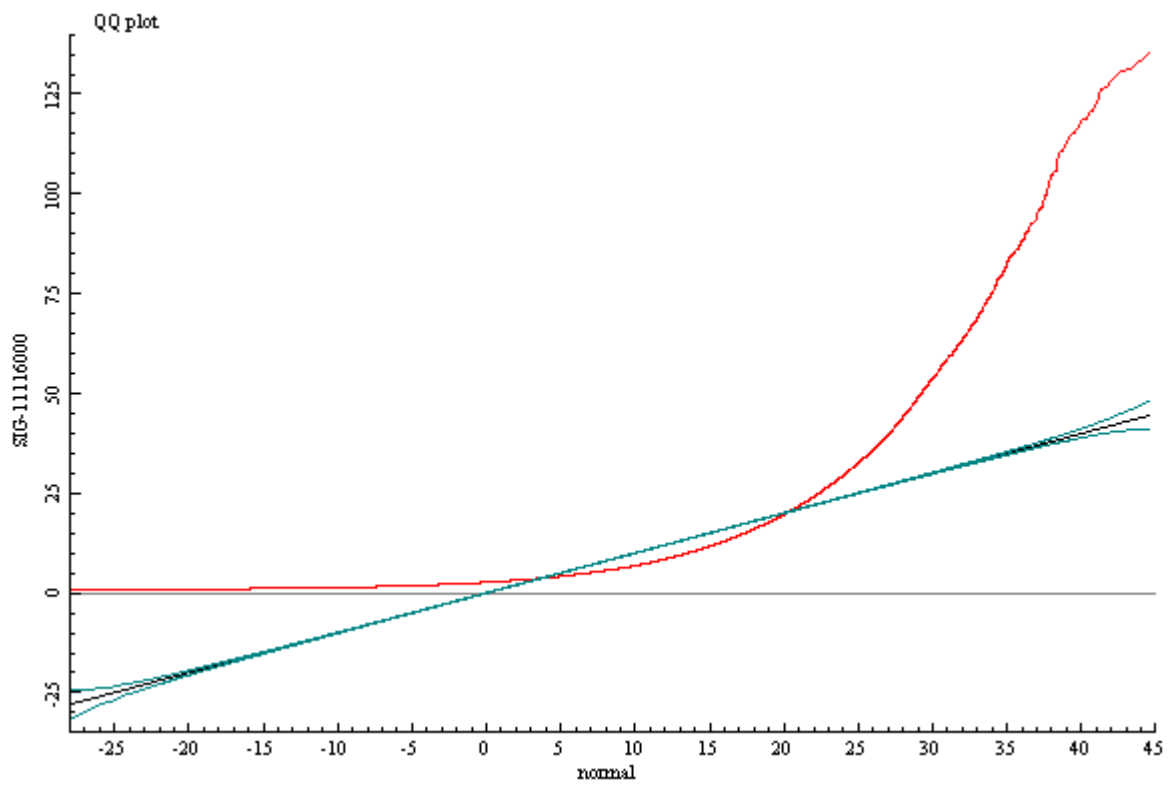
c. One-step-ahead Carbon Conditional Mean QQ-plot



d. One-step-ahead Carbon Conditional Volatility (250 k and 0.5 k)

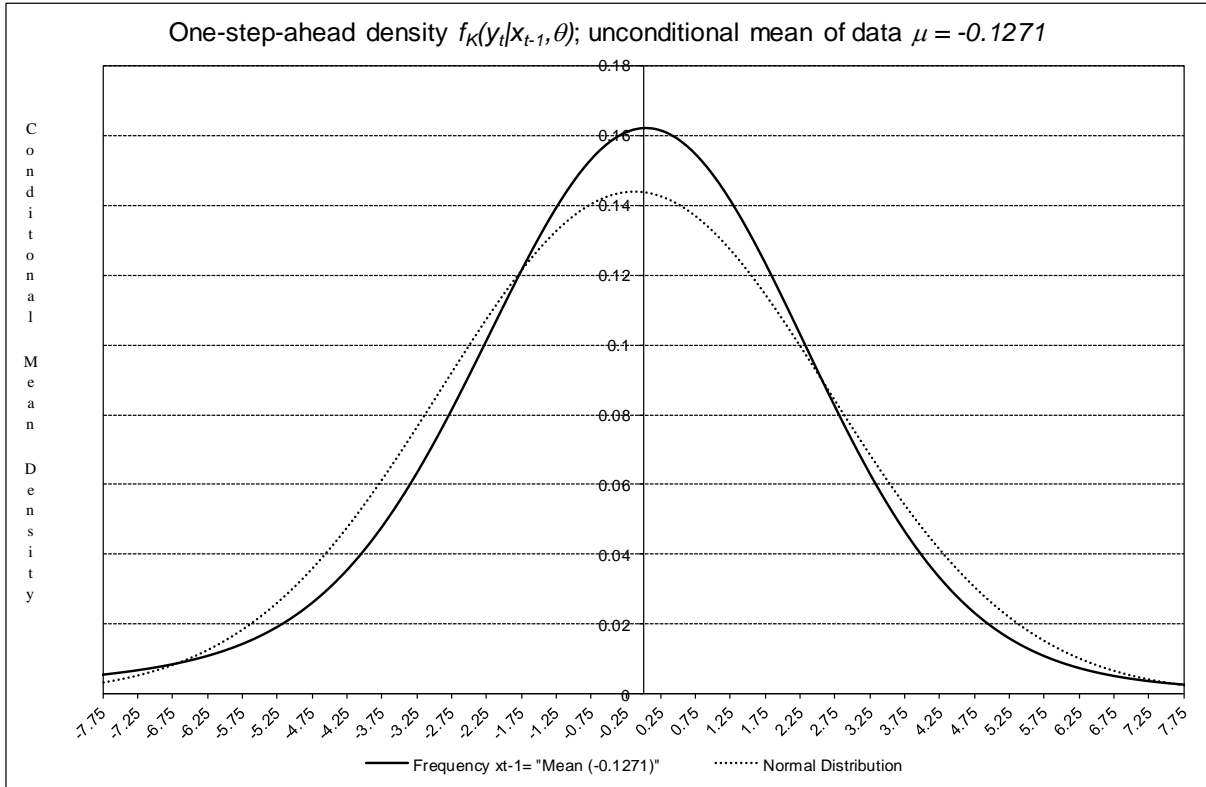


e. One-step-ahead Carbon Conditional Volatility Density

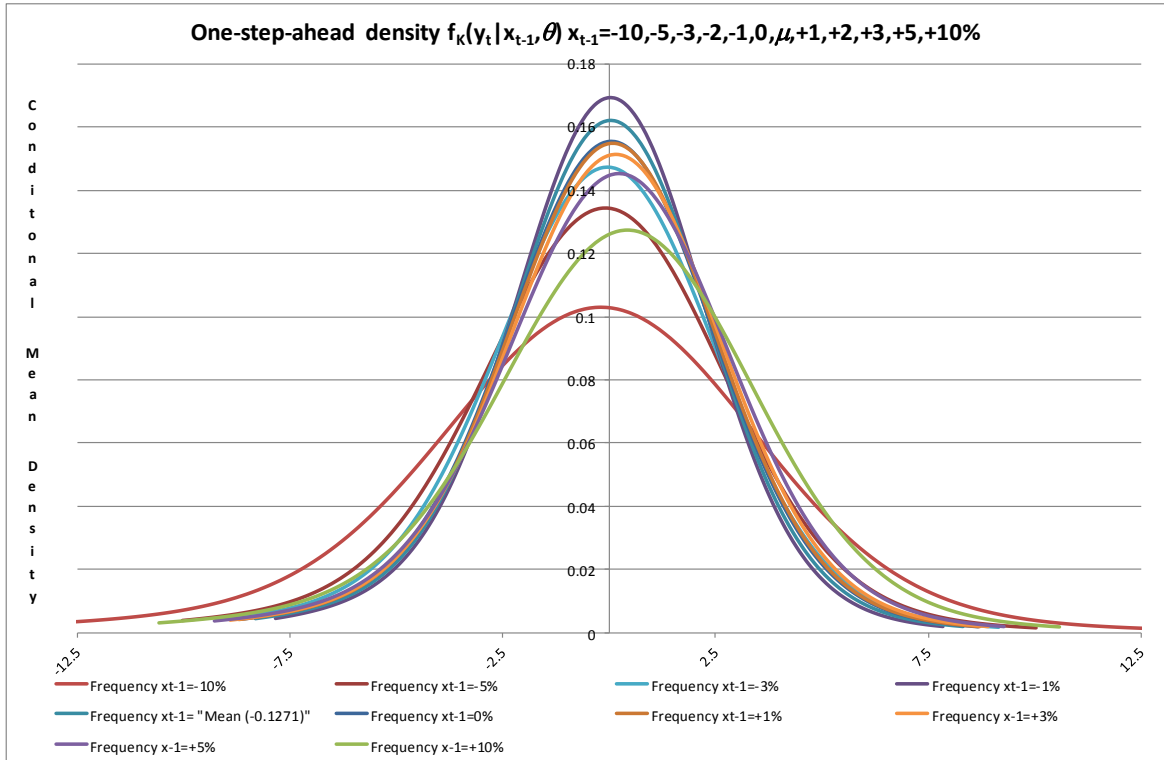


f. One-step-ahead Carbon Conditional Volatility QQ-plot

Figure 8 Carbon Contracts: Conditional mean and volatility from Optimal SV model coefficients

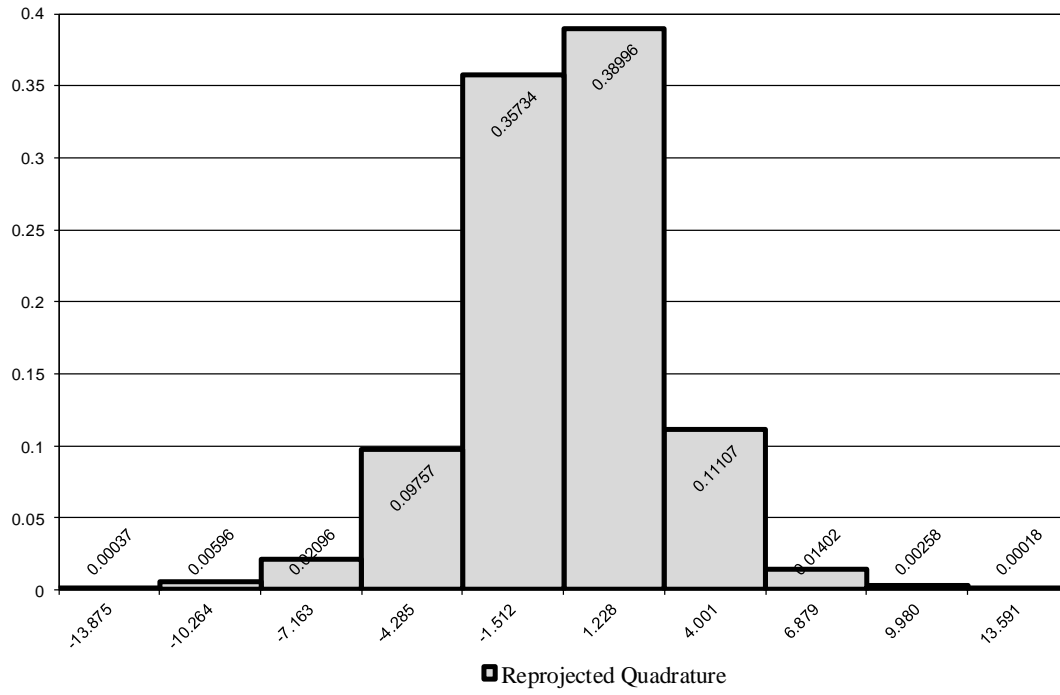


a. One-step-ahead Carbon density (unconditional mean)



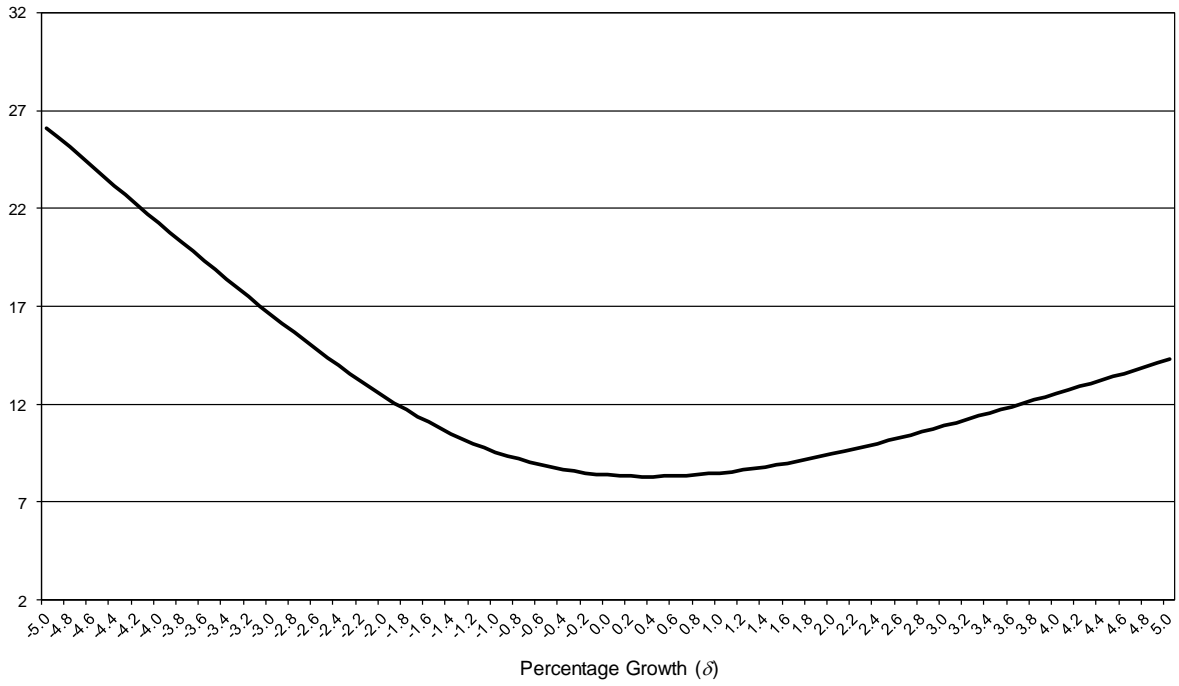
b. One-step-ahead Carbon density (mean: $x_{t-1} = -10\% \dots 10\%$)

GAUSS-hermite Quadrature Density Distribution



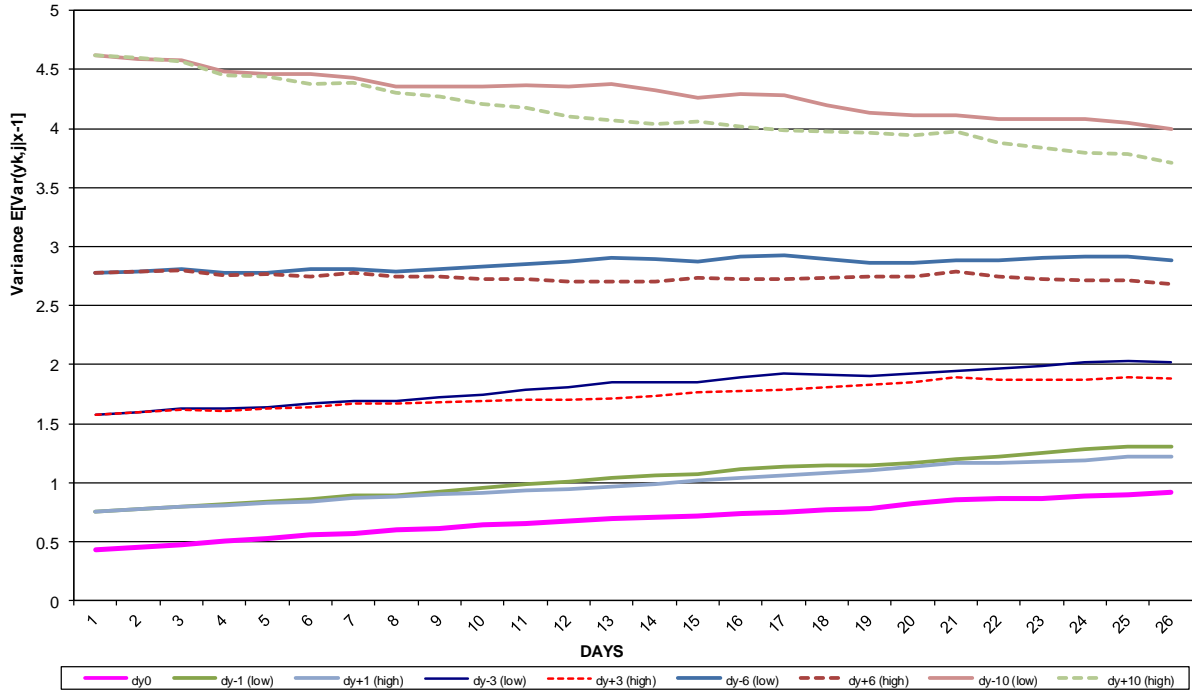
c. Gauss-Hermite Carbon Quadrature

The Conditional Variance Function Front Year Reprojection: the "Assymetry effect"

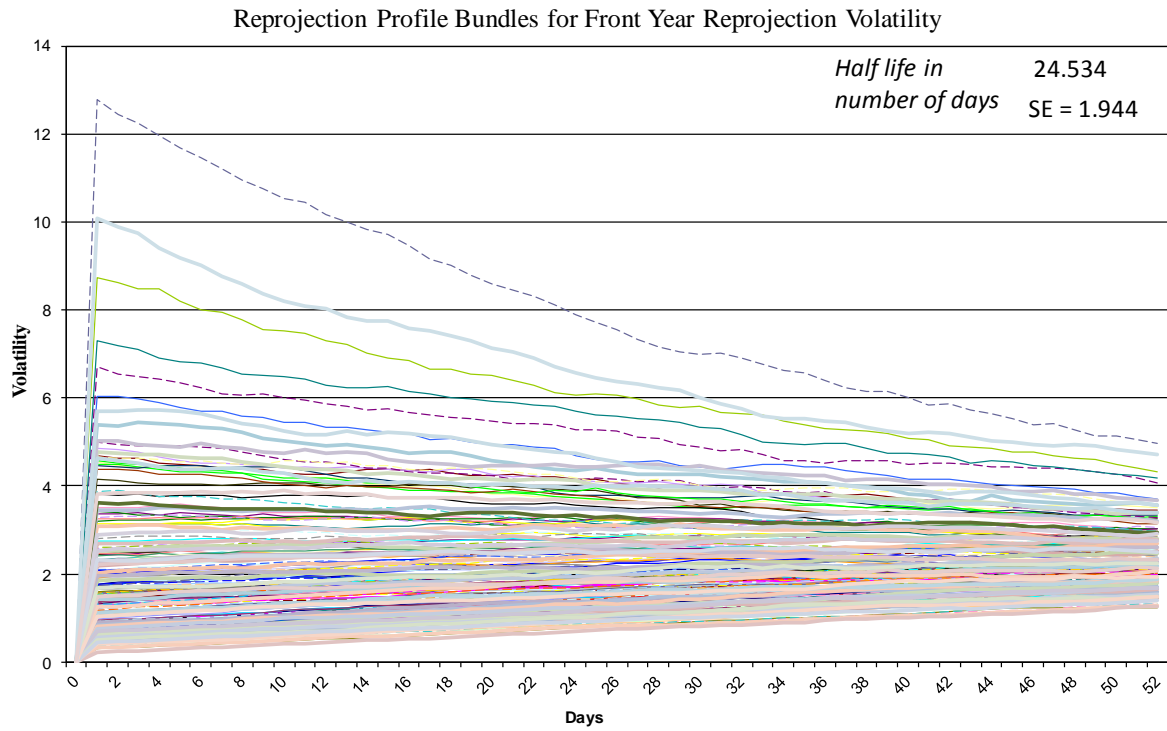


d. The Carbon Conditional Variance Functon

Multistep Ahead Dynamics Front Year Contracts (σ_j^2)



e. Carbon Multistep-ahead Dynamics



f. Profile Bundles for Carbon Volatility persistence

Figure 9 Carbon Contracts: One-Step-Ahead Unconditional (a) and Conditional (b) Mean; Gauss-Hermite Quadrature (c); Conditional Variance Function (d); Multi-Step Ahead Dynamics (e) and Profile Bundles Volatility Persistence (f)