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The Profit-Maximizing Lot Size Problem with Pricing Lag Effects

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Preface

This thesis was written as part of a degree in Master of Science in Logistics, specialization Operation Management at Molde University College. It was supervised by Asmund Olstad and Bård-Inge Pettersen.

I would like to thank my advisors, Olstad and Pettersen, for their guidance in this endeavour, particularly Pettersen for his continuing feedback throughout the process of writing the thesis. I would also like to thank my colleague, Per Kristian Rekdal, for his valuable feedback regarding the structure of the thesis, and my brother-in-law Anders for his insights. Lastly, I would like to thank my late father Helge for being the man he was.

Summary

In this thesis, we present a disaggregated model for a finite horizon profit-maximizing lot size problem. The demand function in each period is a function of price, not only in the same period, but also in the next period. The problem consists of deciding in which period to produce, how much to produce in each production period, and which price to set in each period. The objective is to maximize the total profit, defined as total revenue minus total setup cost, production cost and inventory holding cost.

Three theorems which restrict the amount of possible optimal solutions are proved. A solution is then provided to the production horizon problem, a sub-problem of the main problem. The production horizon problem is then solved with two different models: with and without the lag effect. The optimal solutions of these models are then compared for varying values of some key parameters.

Contents

1 Motivation

In the current landscape of management science, lot sizing problems have garnered considerable interest. Wagner and Whitin's (2004) exact solution algorithm for the Dynamic Lot Size Problem (DLSP) is a cornerstone of this field, and many heuristic solution algorithms have been developed for the related Capacitated Lot Size Problem (CLSP). However, these models all deal with cost minimization, and profit-maximization models have received relatively little attention.

In particular, to the best of our knowledge, there is a lack of research on pricing campaigns and their effect on customers' purchasing decisions, in the context of lot sizing, and we believe that this is an unexplored topic that can be valuable for decision-makers in the real world. We call this effect "the lag effect".

We have chosen to model the lag effect because we believe it to be a useful extension to the Thomas model (Thomas 1970). Thomas already models and solves the lot sizing problem with varying prices, and it stands to reason that a low enough price may attract customers from subsequent time periods. We believe that this "stockup" behaviour is encountered by producers or retailers who run pricing campaigns.

As an example, for grocery stores, running pricing campaigns is commonplace. Prices are often reduced by 20-40%, and the products that are subject to the pricing campaign is typically changed each week or every other week. We argue that the lag effect is present for many products carried by grocery stores, and that a lot sizing model with lag effects would give a better representation of reality than the model of Thomas (Thomas 1970).

We argue that the stockup effect would be most apparent for products with these qualities:

<u>The product can be stored for some time (from one period to another).</u>

For the lag effect to be present, it must make sense for people to want to store it from one period to the next. Highly perishable products, like strawberries or warm store-bought food, cannot be stored for particularly long, and we argue that these types of products do not have much of a lag effect.

<u>Customers buy the product with some regularity.</u>

We argue that sporadic purchases tend to be less affected by pricing. Since the lag effect relies on customers caring about the price, products of this type would have a relatively small lag effect. An example would be cough medicine; we argue that people buy it if they have a cough, and do not think about it otherwise.

<u>Customers who buy the product are price sensitive.</u>

Tying in with the previous point, the lag effect only exists because people want to save money by buying when the price is low. Therefore, for the lag effect to be present, the customer base must be at least somewhat price sensitive.

We argue that frozen pizza, such as Grandiosa and soda, such as Coca Cola, are good candidates for price campaigns taking the lag effect into account. These products can be stored for weeks or months, and customers tend to buy them regularly. We also argue that the demand of these products is price elastic, as goods with many substitutes typically are (Pindyck and Rubinfeld 2013). In other words, their customers are price sensitive.

For a time period with much higher demand than production capacity, the original Profit Maximizing Capacitated Lot Size Problem (PCLSP) model (Haugen, Olstad and Pettersen 2007) has two ways to deal with the high demand. Either reschedule some or all production to other periods, or raise prices in order to "price out" demand. The lag effect model introduces a third way to deal with this issue.

The lag effect allows the producer to use price as a tool, not only to price out demand, but also to move it from one time period to another. Having a particularly low price, i.e. running a price campaign, in some period will attract customers from the next period. These customers will then buy two units in the period of the price campaign, rather than one unit and then another unit in the next period.

These are the research questions of this thesis:

- Can any of Thomas' lemmas (Thomas 1970) be adapted to our model? If so, which ones?
- Can the model be solved, either exactly or heuristically?
- How does the optimal solution of the model, for a given data set, behave compared to that of the Thomas model (Thomas 1970)?

2 Background

Traditionally in operation management literature, internal production processes have been managed independently from marketing decisions. In this case, pricing is decided first, and subsequently the production of the resulting demand is optimized with respect to cost.

Breaking away from this trend, Thomas (1970) replaces fixed demand with demand functions, thus making the transition from cost minimization to profit maximization in a monopoly market. Thomas gives a method for quickly determining optimal prices, in particular for linear demand functions. Thomas also gives a solution algorithm to this new model, a modification of the Wagner-Whitin algorithm (Wagner and Whitin 2004).

In a review of CLSP algorithms, Karimi and Wilson (2003) list 38 different algorithms for solving the traditional CLSP, some heuristically, and some to optimality. These algorithms include improvement heuristics, relaxation heuristics, branch-and-bound heuristics, as well as other types.

For profit-maximizing models, Geunes, Romeijn and Taaffe (2006) model a single-item profit-maximizing lot size problem with a non-varying capacity in each period. They then provide solutions in polynomial time, to the problem with piecewise linear or concave revenue functions. Gonzalez-Ramirez, Smith and Askin (2011) model a multi-product profit-maximizing CLSP with setup times and present a heuristic that can solve large problem instances quickly.

Haugen, Olstad and Pettersen (2007) solve a similar problem, but without setup times, heuristically, using a Lagrangian relaxation algorithm. Lanquepin-chesnais, Haugen and Olstad (2012) develop a heuristic algorithm for the same problem, without setup times, but with prices only chosen from a finite, discrete set.

The Thomas algorithm (Thomas 1970) is a key part of the heuristic algorithm of Haugen, Olstad and Pettersen (2007) for the PCLSP. Our intention is for our algorithm for the Thomas problem with lag effect to be used as a step in the PCLSP algorithm in place of the original Thomas algorithm.

The algorithm for the PCLSP (Haugen, Olstad and Pettersen 2007) is based on Lagrangian relaxation (see figure 1 for an overview). It divides the main problem into two sub-problems; the Lagrangian Upper Bound Problem (LUBP) and the Lagrangian Lower Bound Problem (LLBP). In the article of Haugen, Olstad and Pettersen (2007), the LUBP is a single-product, uncapacitated version of the main problem. Haugen, Olstad and Pettersen solve this sub-problem with the Thomas algorithm (Thomas 1970).

For a PCLSP with the lag effect, the corresponding LUBP would also include the lag effect. This sub-problem would then need a modified version of the Thomas algorithm to solve, which is what we aim to develop in this thesis. Like Thomas' model (1970), our model assumes a monopoly market.

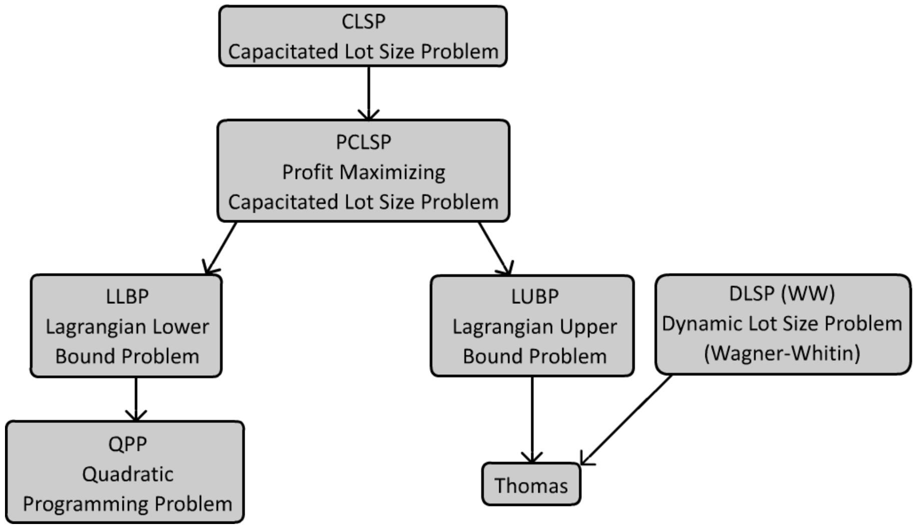

Figure 1: PCLSP algorithmic overview.

The most basic of the models we base this thesis on is the Dynamic Lot Size Problem (Wagner and Whitin 2004). It concerns a finite planning horizon with given demands of a single product, inventory holding costs and setup costs, and aims to minimize the total cost while satisfying demand. Wagner and Whitin (2004) developed a exact solution algorithm for this problem, based on dynamic programming.

Wagner and Whitin's algorithm utilizes five theorems to reduce the computational effort needed:

- Theorem 1: It is optimal to never carry inventory into a period of production.
- Theorem 2: It is optimal to have each nonzero production quantity be a sum of demands.
- Theorem 3: If demand in some period is satisfied by production in an earlier period, then so is the demand of all periods between the two.
- Theorem 4: If there is no incoming inventory in some period t, it is optimal to consider periods $1...t$ by themselves.
- Theorem 5, the Horizon Theorem: If, for some sub problem $1...t^*$ it is optimal to have the final production in period t^{**}, $t^{**} < t^*$, then for later sub problems, $1...t$, it is sufficient to consider only periods $t^{**}...t$.

The Thomas algorithm (Thomas 1970) is also based on these 5 theorems, and Thomas proves in his paper that these theorems hold true for the Thomas model. Theorems 1-3 allow Thomas to describe the problem as a dynamic programming problem, while theorems 4 and 5 help cut down on the time required to compute each step, especially with a high number of periods.

It is our expectation that theorems 1-3 will hold true for our lag effect model, but that theorems 4 and 5 will not hold true. If this is indeed the case, we may still be able to use a dynamic programming approach, but there will be no obvious opportunity to cut down on the problem size of each step.

3 Formulations

3.1 The stockup lag effect

The stockup lag effect stems from the idea that if the price of a product is low enough, some customers may choose to stock up on said product. These customers will then abstain from purchasing the product in the "lag" time, at some later point where they would otherwise have wanted to make a purchase.

In the context of lot sizing, we model this as demand being transferred from a period to the period before it. The amount of demand transferred is determined by a lag function, which is dependent on the price in the first of the two periods. Although a lag effect lasting more than one period is definitely conceivable, we limit the scope of this thesis to consider lag effects on one period. In other words, setting a low price will only attract customers from the next period, and not from subsequent ones.

Formally, we describe the lag function as follows:

$$l_t(p_t) = \text{demand transferred from period } t+1 \text{ to period } t \text{ through the stockup lag effect}$$

where p_t is the price in period t.

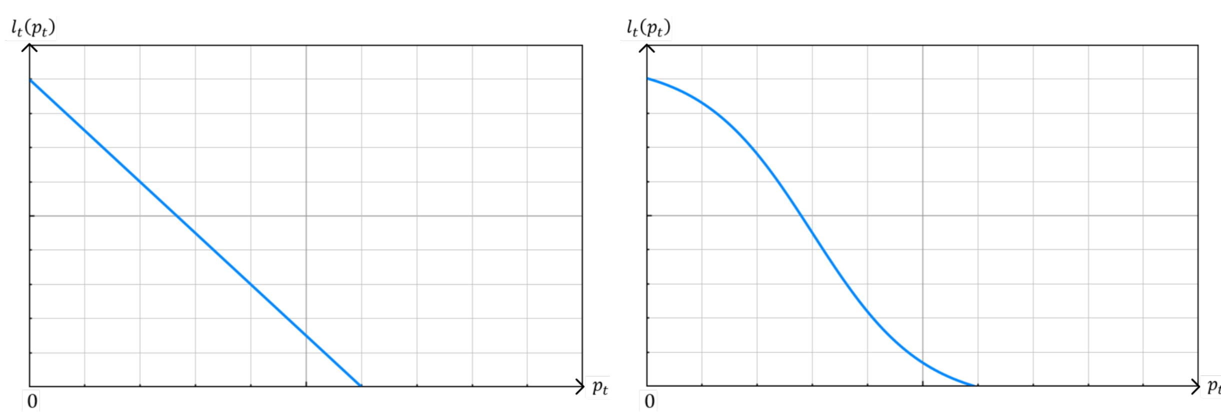

Figure 2: Two examples of lag functions. These are sketches to illustrate the point that the lag function needs not be linear.

In figure 2, we see plots of two different lag functions. In the lag function on the left, the customers' sensitivity to pricing in the previous period is the same no matter if the price is high or low. This lag function presents as a straight line.

In the lag function on the right, customers are less sensitive to pricing in the previous period if that price is already high or low, and more sensitive if the price is around the middle of the graph. This presents as a graph that is steeper around the middle than on the edges. These are only two examples of many possible lag functions.

A demand function for period t with the stockup lag effect looks as follows:

$$d_t = \phi_t(p_t) + l_t(p_t) - l_{t-1}(p_{t-1}). \tag{1}$$

where

$$\phi_t(p_t) = \text{component of the demand function in period } t \text{ independent of the lag effect.}$$

Note the two terms with lag functions. The term with the positive sign is demand transferred from period $t+1$ to period t, while the negative term represents demand going from period t to period $t-1$. In other words, the function $l_t(p_t)$ represents what the period receives from the next period. Note also that, due to the lag effect, d_t is a function of two variables: the price in the same period, p_t, and in the previous period, p_{t-1}.

3.2 Assumptions

We make some assumptions on the model functions and parameters, both for reasons of economical interpretation and for the existence of a solution. We consider problems that obey the following conditions:

Assumption 1:

The lag function is non-increasing with the lag price.

$$\frac{\partial l(p_t)}{\partial p_t} \leq 0 \; \forall \; t = 1, ..., T. \tag{2}$$

This assumption says that, as the price in some period increases, less (or actually "not more") demand will be transferred to it from the next period. In other words, increasing the price will lead to fewer customers making a purchase in this period rather than the next.

Assumption 2:

The lag-independent demand function is non-increasing with price.

$$\frac{\partial \phi(p_t)}{\partial p_t} \leq 0 \; \forall \; t = 1, ..., T. \tag{3}$$

Similar to the previous condition, this assumption says that, as the price in some period increases, the period's lag-independent demand will decrease. This means that some potential customers will find the price too high and instead spend their money on substitute goods.

Assumptions 3-4:

The lag function and the lag-independent demand function are non-negative.

$$l_t(p_t)\ ,\ \phi_t(p_t) \geq 0\ \forall\ t = 1, ..., T. \tag{4}$$

These assumptions also stem from the economic interpretation of the functions. A non-negative lag function $l_t(p_t)$ means that demand can be transferred backwards in time (from period $t+1$ to t), but not forwards (t to $t+1$). This assumption is also the reason we call it the "stockup" lag effect. We argue that a customer who abstains from purchasing in some period will not be "twice as hungry" for the product in the next period.

A non-negative lag-independent demand function $\phi_t(p_t)$ means that, out of the lag-independent demand in some period, less than zero of this demand cannot be realised. In other words, these customers cannot sell anything back to the producer.

Assumptions 5-6:

The lag function and the lag-independent demand function are continuously differentiable.

$$\frac{\partial l_t(p_t)}{\partial p_t}\ ,\ \frac{\partial \phi_t(p_t)}{\partial p_t}\ \textit{continuous}. \tag{5}$$

These assumptions have to do with the solution of the problem. As we will see later, we are able to solve a simplified variant of the problem (the production horizon problem) using the derivatives of these two functions.

Assumption 7:

The Hessian matrix of the objective function Π *is negative semidefinite.*

This assumption also has to do with the solution of the problem. It ensures that the problem does indeed have an optimal solution. If the Hessian matrix is negative definite, then the unconstrained objective function has exactly one maximum. A negative semidefinite Hessian may have multiple maxima (Eriksen and Gustavsen 2010).

3.3 The disaggregated Thomas model extended with stockup lag effect

We have chosen to use a disaggregated model instead of an aggregated one. In an aggregated model, we have one production variable (let us call it x_t) and one inventory variable (I_t) for each time period. These variables show how much product is produced in each period, and how much is kept in inventory until the next period.

In a disaggregated model, on the other hand, we have production-demand variables, which we call d_{it}. These variables show how much is produced in period i to cover demand in period t. The total number of variables is increased in a disaggregated model, but this also allows for a more detailed model.

Based on the model by Thomas (1970), the extended model is as follows:

$$\max \Pi = \sum_{t=1}^{T} \left[d_t p_t - s_t \delta_t - \sum_{i=t}^{T} h_{ti} d_{ti} \right], \tag{6}$$

subject to

$$d_t = \phi_t(p_t) + l_t(p_t) - l_{t-1}(p_{t-1}) \quad \forall\, t = 1, ..., T, \tag{7}$$

$$\sum_{i=1}^{t} d_{it} = d_t \quad \forall\, t = 1, ..., T, \tag{8}$$

$$\sum_{i=t}^{T} d_{ti} \leq M_t \delta_t \quad \forall\, t = 1, ..., T, \tag{9}$$

$$l_0(p_0) = l_0 \tag{10}$$

$$l_T(p_T) = l_T, \tag{11}$$

$$p_t, d_{it} \geq 0 \quad \forall\, i = 1, ..., T,\; t = i, ..., T, \tag{12}$$

$$\delta_t \in \{0, 1\} \quad \forall\, t = 1, ..., T. \tag{13}$$

Variables and functions:

$\delta_t = 1$ if production occurs in period t ; 0 otherwise
$p_t =$ price in period t
$d_{it} =$ amount produced in period i to cover demand in period t
$d_t =$ demand in period t
$l_t(p_t) =$ demand transferred from period $t+1$ to period t through the stockup lag effect
$\phi_t(p_t) =$ component of the demand function in period t independent of the lag effect

Parameters:

$$
\begin{aligned}
T &= \text{amount of time periods in planning horizon} \\
s_t &= \text{setup cost in period } t \\
c_t &= \text{cost of producing one unit in period } t \\
h_t &= \text{cost of holding one unit as inventory from period } t \text{ to period } t+1 \\
h_{ti} &= c_t + \sum_{k=t}^{i-1} h_k = \text{cost of producing one unit in period } t \text{ and} \\
&\quad \text{holding it in inventory until period } i \\
M_t &= \text{upper limit on production in period } t \\
l_0 &= \text{lag effect constant between periods 0 and 1} \\
l_T &= \text{lag effect constant between periods T and T+1}
\end{aligned}
$$

Note that h_{it} includes both production costs and inventory holding costs. h_{ii} includes only production costs, as it is produced and sold in the same period, and not held in inventory.

Objective function and constraints:

Eq.(6) is the objective function. It describes total profit as total revenue minus total setup cost, production cost and inventory holding cost.

Eq.(7) describes the demand function as a sum of its components; the lag effect and the the part independent of the lag effect.

Eq.(8) says that the demand function is the sum of its disaggregated variables.

Eq.(9) says that no production can take place in a period without setup.

Eq.(10) sets the lag effect between periods 0 and 1 to a given constant.

Eq.(11) does the same for the lag between periods $T+1$ and T.

Eq.(12) ensures non-negativity for price and production-demand variables, while Eq.(13) ensures that the setup variables are binary.

4 Some analytical results

4.1 The Thomas theorems

The Thomas model (Thomas 1970) relies on a number of lemmas and theorems in order to restrict the amount of potential solutions and decrease the time it takes to obtain the optimal solution. Some of these theorems can be adapted to our model with the lag effect, while others cannot. Most notably, the Horizon theorem cannot be adapted to the lag effect in any obvious way.

In this subsection we will adapt some of Thomas' (1970) lemmas to our model with disaggregated variables and lag functions. For each lemma, we will first state the meaning, and then follow with a proof.

4.1.1 Extension of lemma 1

Lemma 1 of Thomas (1970) states that it is optimal never to have inventory going into a period of production.

Lemma 1:

If there exists an optimal solution to (6) - (13), then there exists an optimal solution where

$$d_{it} \cdot d_{tj} = 0 \quad \forall\ t = 1, ..., T,\ i = 1, ..., t-1,\ j = t, ..., T. \tag{14}$$

Proof:

Assume there is an optimal solution where $d_{it} > 0$ and $d_{tj} > 0$ for some given i, t and j. In this solution, we call the set of periods before t, in which production took place: $W = \{k | d_{kl} > 0, k \in \{1, ..., t-1\}, l \in \{k, ..., T\}\}$. The cost of producing a unit of product in period k and keeping it in inventory until period t is h_{kt}.

Since we assume that the solution is optimal, d_{it} must have been produced in the cheapest period in W with regard to production costs and inventory holding costs, i.e. $i = \text{argmin}\{h_{kt} | k \in W)\}$.

If $h_{it} > h_{tt}$, then it is more costly to produce in period i for period t than it is to produce in period t for the same period. In this case, rescheduling $d_{it} + d_{tj}$ to be produced in period t will be cheaper, and the original solution cannot be optimal.

If $h_{it} < h_{tt}$, then it is more costly to produce in period t for the same period, than to produce in period i for period t. Then, rescheduling $d_{it} + d_{tj}$ to be produced in period i and held as inventory until period t will be cheaper, and the original solution cannot be optimal.

If $h_{it} = h_{tt}$, then by rescheduling all production of period t to period i, one can save the setup cost of period t, s_t, and the original solution cannot be optimal.[1] That is, let $\delta_t = 0$, and for each $m = t, ..., T$, let

$$\hat{d}_{im} = d_{im} + d_{tm}, \tag{15}$$

$$\hat{d}_{tm} = 0, \tag{16}$$

where $\hat{d}_{im}$ is the updated value.

4.1.2 Extension of lemma 2

Lemma 2 of Thomas (1970) states that it is optimal to have each nonzero production quantity be a sum of demands.

Lemma 2:

If there exists an optimal solution to (6) - (13), then there also exists an optimal solution such that, for each t, there exists an $i \leq t$ such that $d_{it} = d_t$.

Proof:

Assume there is an optimal solution where the above does not hold. $d_{it} > d_t$ for any $i \leq t$ would break constraint (8), so this would be infeasible. This means that $0 \leq d_{it} < d_t \ \forall \ i \leq t$.

d_{it} does not cover all demand in period t, as $d_{it} < d_t \ \forall \ i \leq t$. This means that the production to cover demand in period t is split into two or more periods. Of these periods in which production takes place, let us denote the next-to-latest of which period j, $j < t$, and the latest period k, $j < k < t$.

As $d_{jt} > 0$, some amount of product is held in inventory from period j to t via period k, that is, at least d_{jt} is going into period k as inventory. This contradicts lemma 1, and the solution cannot be optimal.

[1] If setup costs of 0 are allowed, then then this procedure may yield an alternative optimal solution. If not allowed, then an optimal solution must obey Eq.(14)

4.1.3 Extension of lemma 3

Lemma 3 of Thomas (1970) states that if demand in some period t is satisfied by production in some period i, $i < t$, then each period j, $i < j < t$ is also satisfied by period i.

Proof:

Suppose that for periods $i < j < t$, period t is satisfied by period i, but period j is *not* satisfied by period i, and that the solution is optimal. Period j must then have been satisfied by some other period k, $k < j$, $k \neq i$.

If $k < i$, then d_{kj} must be held as inventory going into period i, contradicting lemma 1, and the solution cannot be optimal. If, on the other hand, $k > i$, then d_{it} must be held as inventory going into period k, contradicting lemma 1, and the solution cannot be optimal.

4.2 The extended production horizon problem

Using the lemmas from the previous subchapter, in particular lemma 2 and 3, we are now ready to look at the production horizon problem. The planning horizon is an important sub-problem for the Thomas algorithm (Thomas 1970), in which Thomas solves the production horizon problem to optimality. The Thomas model is solvable as a sum of these individual production horizon problems. We want to investigate whether this variant of the Thomas model with lag effects is solvable as a function of these production horizons.

4.2.1 Formulation

The production horizon problem concerns a finite planning horizon, where setup occurs only in period 1. Production from period 1 is held in inventory to satisfy demand in every period in the planning horizon. The objective is to decide the price in each period in order to maximize total profit.

Since all demand in each period in the production horizon is satisfied by production in period 1, the contribution to total profit from period t can be written as

$$\gamma_t = d_{1t}p_t - h_{1t}d_{1t}. \tag{17}$$

The total profit in a production horizon can be written as a sum of these components:

$$\max \pi = \sum_{t=1}^{T} \gamma_t, \tag{18}$$

subject to

$$\gamma_t = d_{1t}p_t - h_{1t}d_{1t} \quad \forall\ t = 1, ..., T, \tag{19}$$

$$d_{1t} = \phi_t(p_t) + l_t(p_t) - l_{t-1}(p_{t-1}) \quad \forall\ t = 1, ..., T, \tag{20}$$

$$l_0(p_0) = l_0, \tag{21}$$

$$l_T(p_T) = l_T, \tag{22}$$

$$p_t, d_{1t} \geq 0 \quad \forall\ t = 1, ..., T. \tag{23}$$

4.2.2 Solution

The objective function for the production horizon is as follows:

$$\begin{aligned}
\pi &= \sum_{t=1}^{T} \gamma_t, && (24)\\
&= \sum_{t=1}^{T} \Big[\phi_t(p_t) + l_t(p_t) - l_{t-1}(p_{t-1})\Big](p_t - h_{1t}), && (25)\\
&= \sum_{t=1}^{T} \Big[\phi_t(p_t)p_t + l_t(p_t)p_t - l_{t-1}(p_{t-1})p_t - \phi_t(p_t)h_{1t} - l_t(p_t)h_{1t} + l_{t-1}(p_{t-1})h_{1t}\Big]. && (26)
\end{aligned}$$

Differentiating this with respect to $p_t,\ t = 1, ..., T$, we obtain, for period 1:

$$\begin{aligned}
\frac{\partial \pi}{\partial p_1} = \frac{\partial \phi_1(p_1)}{\partial p_1}p_1 + \phi_1(p_1) + \frac{\partial l_1(p_1)}{\partial p_1}p_1 + l_1(p_1)\\
- \frac{\partial \phi_1(p_1)}{\partial p_1}h_{11} - \frac{\partial l_1(p_1)}{\partial p_1}h_{11} - l_1(p_1)p_2,
\end{aligned} \tag{27}$$

for periods $2, ..., T-1$:

$$\begin{aligned}
\frac{\partial \pi}{\partial p_t} = \frac{\partial \phi_t(p_t)}{\partial p_t}p_t + \phi_t(p_t) + \frac{\partial l_t(p_t)}{\partial p_t}p_t + l_t(p_t) - l_{t-1}(p_{t-1})\\
- \frac{\partial \phi_t(p_t)}{\partial p_t}h_{1t} - \frac{\partial l_t(p_t)}{\partial p_t}h_{1t} - l_t(p_t)p_{t+1} \quad \forall\ t = 1, ..., T,
\end{aligned} \tag{28}$$

and for period T, we obtain:

$$\begin{aligned}
\frac{\partial \pi}{\partial p_T} = \frac{\partial \phi_T(p_T)}{\partial p_T}p_T + \phi_T(p_T) + \frac{\partial l_T(p_T)}{\partial p_T}p_T + l_T(p_T) - l_{T-1}(p_{T-1})\\
- \frac{\partial \phi_T(p_T)}{\partial p_T}h_{1T} - \frac{\partial l_T(p_T)}{\partial p_T}h_{1T}.
\end{aligned} \tag{29}$$

These equations can each be set equal to zero, and the resulting equation system will give an optimal solution to the planning horizon problem.

The solution to the planning horizon problem yields T equations with T variables, a $T \times T$ equation system. For linear lag functions, this is easily solvable, especially if using a specialized algorithm. For more complicated lag functions, however, this may be more difficult.

4.2.3 The case of linear demand functions and lag functions with two periods

Let us look at a simple example where:

$$T = 2, \qquad \text{(two periods)} \tag{30}$$

$$\phi_t = \alpha_t - \beta_t p_t, \qquad t = 1, 2 \tag{31}$$

$$l_1 = \alpha_2 - \beta_2 p_1. \tag{32}$$

We assume as always that the boundary lag effects l_0 and l_2 are zero. The profit function is given by:

$$\begin{aligned} \pi &= \gamma_1 + \gamma_2, & (33)\\ &= d_1(p_1 - h_{11}) + d_2(p_2 - h_{12}), & (34)\\ &= (\phi_1 + l_1)(p_1 - h_{11}) + (\phi_2 - l_1)(p_2 - h_{12}). & (35) \end{aligned}$$

We then get the first order optimality condition with regard to p_2:

$$\begin{aligned} \pi_{p_2} &= 0 + \phi_2 - l_1 + (\phi_{2p_2} + 0)(p_2 - h_{12}), & (36)\\ &= \alpha_2 - \beta_2 p_2 - \alpha_2 + \beta_2 p_1 - \beta_2 p_2 + \beta_2 h_{12}, & (37)\\ &= \beta_2\big(p_1 - (2p_2 - h_{12})\big), & (38)\\ &= 0, & (39) \end{aligned}$$

or

$$p_2 = \frac{1}{2}(p_1 + h_{12}). \tag{40}$$

We then get the first order optimality condition with regard to p_1:

$$\begin{aligned} \pi_{p_1} &= \phi_1 + l_1 + (\phi_{1p_1} + l_{1p_1})(p_1 - h_{11}) - l_{1p_1}(p_2 - h_{12}),\\ &= \alpha_1 - \beta_1 p_1 + \alpha_2 - \beta_2 p_1 - (\beta_1 + \beta_2)(p_1 - h_{11}) + \beta_2(p_2 - h_{12}) = 0. & (41) \end{aligned}$$

Observe that both first order conditions depend on both p_1 and p_2. To solve this equation system, we need to substitute one equation into another, twice.

Substituting Eq.(40) into Eq.(41) we get:

$$\alpha_1 - \beta_1 p_1 + \alpha_2 - \beta_2 p_1 - (\beta_1 + \beta_2)(p_1 - h_{11}) + \beta_2(\frac{1}{2}(p_1 + h_{12}) - h_{12}) = 0, \tag{42}$$

$$\alpha_1 - \beta_1 p_1 + \alpha_2 - \beta_2 p_1 - (\beta_1 + \beta_2)(p_1 - h_{11}) + \frac{1}{2}\beta_2(p_1 - h_{12}) = 0. \tag{43}$$

Solving Eq.(43) with regard to p_1 we get an expression for p_1:

$$p_1 = \frac{1}{2\beta_1 + \frac{3}{2}\beta_2}\Big(\alpha_1 + \alpha_2 + h_{11}(\beta_1 + \beta_2) - \frac{1}{2}\beta_2 h_{12}\Big). \tag{44}$$

Substituting Eq.(44) into Eq.(40) we get an expression for p_2:

$$\begin{aligned} p_2 &= \frac{1}{2}(p_1 + h_{12}), && (45)\\ &= \frac{1}{2}\Big(\frac{1}{2\beta_1 + \frac{3}{2}\beta_2}\Big(\alpha_1 + \alpha_2 + h_{11}(\beta_1 + \beta_2) - \frac{1}{2}\beta_2 h_{12}\Big) + h_{12}\Big), && (46)\\ &= \frac{1}{4\beta_1 + 3\beta_2}\Big(\alpha_1 + \alpha_2 + h_{11}(\beta_1 + \beta_2) - \frac{1}{2}\beta_2 h_{12}\Big) + \frac{1}{2}h_{12}. && (47) \end{aligned}$$

Verified by a solver for a specific data set, these equations do indeed yield the optimal prices for the 2-period production horizon. For the programmed mathematical model, parameter values and solution, view appendix A.

5 Analysis and discussion

In this chapter we will look at how the model acts when compared to the original Thomas model (Thomas 1970). More specifically, we will solve the production horizon problem, both for the original Thomas model, and the Thomas model with the lag effect. We will do this by using MINOS, a commercially available solver for problems with quadratic objective functions. For the model of the production horizon problem, see Eq.(17) - (23).

The purpose of this is to gain some knowledge regarding how the lag model behaves when compared to the Thomas model (Thomas 1970). For example, which model generally gives the highest profit? How large is the deviation usually, in percentage and in absolute value? We will then attempt to explain the results.

5.1 Demand function and lag function

For the sake of simplicity, we choose to use linear demand functions and lag functions in this analysis. We define the demand function as follows:

$$d_t = \phi(p_t) + l_t - l_{t-1} \quad \forall\ t = 1, ..., T, \tag{48}$$

where

$$\begin{aligned} \phi_t(p_t) &= \alpha_t - \beta_t p_t \\ \alpha_t &= \text{base demand in period } t\ ^2 \\ \beta_t &= \text{slope in demand function of period } t \\ d_t, p_t, l_t &= \text{as described earlier.} \end{aligned} \tag{49}$$

The lag function is defined as follows:

$$l_t = f\phi_{t+1}(p_t) \quad \forall\ t = 1, ..., T-1, \tag{50}$$

$$0 \leq f \leq 1, \tag{51}$$

where

$f =$ the fraction of demand that may transfer to the previous period due to low price, let us call it the *willingness to stock up.*

[2]"Base demand" is what we call the amount that would be purchased for consumption in the same period, if the price were zero.

For a customer to stock up on product, two independent criteria both need to be fulfilled: the customer must be *willing to stock up*, and *willing to pay* the price. Observe, in Eq.(51), that these two effects are both represented. The willingness to stock up is, of course, represented by f, and the willingness to pay is represented by $\phi_{t+1}(p_t)$.

We let the boundary lag effects be zero:

$$l_0 = l_T = 0. \tag{52}$$

5.2 Input and output

For the analysis, we will be varying three key parameters: The f factor belonging to the lag functions, the base demand, α_t, and the amount of time periods, T. For simplicity, we will have all α_t equal,

$$\alpha_t = \alpha \quad \forall\ t = 1, ..., T, \tag{53}$$

regardless of time period t, and also all β_t equal,

$$\beta_t = \beta \quad \forall\ t = 1, ..., T. \tag{54}$$

Economically, we can interpret the non-variance in α and β as a stable market. Customers do not enter and leave the market, and customers are equally price-sensitive in all periods.

We will only change one of these parameters, f, α or T, at a time. For each value of the parameter, we will find the profit of each of the two models, as well as the difference between them, in both percentage of the lowest, and in absolute value.

Let us now define some functions with which we will interpret the results:

$$\pi_{LAG} = \text{Optimal total profit of the lag model for a given instance} \tag{55}$$

$$\pi_{THOM} = \text{Optimal total profit of the Thomas model for a given instance} \tag{56}$$

$$D = \pi_{LAG} - \pi_{THOM} = \text{Difference between lag profit and Thomas profit} \tag{57}$$

$$Q = \Big(\frac{\pi_{LAG}}{\pi_{THOM}} - 1\Big) \cdot 100\% = \text{difference between the lag profit and Thomas profit, in }\% \tag{58}$$

5.3 Instances

For the initial instance, we choose our data as follows:

$$\alpha = 10, \qquad \beta = 1, \tag{59}$$
$$T = 4, \qquad f = 1, \tag{60}$$
$$h_{11} = 2, \qquad h_{12} = 3, \tag{61}$$
$$h_{13} = 4, \qquad h_{14} = 5. \tag{62}$$

This will be our starting point from which we will change one parameter at a time.

For variations in α, we will use values of 1, 2, 3, 4, and then in increments of 2 up to 30. The factor f will be used in increments of 0.1 from 0.0 to 1.0. Time periods T will be used in increments of 1 from 1 to 9.

When increasing the amount of time periods T, the unit production cost plus inventory holding cost will be calculated as

$$h_{1t} = 1 + t \quad \forall\ t = 1, ..., T, \tag{63}$$

in other words, producing for period 1 will cost 2, producing for period 2 will cost 3 and so on.

5.4 Results

Below are the numerical results of varying α.

α	π_{LAG}	π_{THOM}	D	Q
1	0	0	0	-
2	0	0	0	-
3	0.50	0.25	0.25	100%
4	2.13	1.25	0.88	70.67%
6	10.71	7.50	3.21	42.74%
8	27.73	21.50	6.23	28.99%
10	53.53	43.50	10.03	23.05%
12	88.14	73.50	14.64	19.92%
14	131.59	111.50	20.09	18.02%
16	183.86	157.50	26.36	16.73%
18	244.95	211.50	33.45	15.81%
20	314.87	273.50	41.37	15.12%
22	393.61	343.50	50.11	14.59%
24	481.18	421.50	59.68	14.16%
26	577.57	507.50	70.07	13.81%
28	682.78	601.50	81.28	13.51%
30	796.82	703.50	93.32	13.27%

Figure 3: Solutions for variation in α. Two optimizations are run for each row in the table: one *with* the lag effect (π_{LAG}) and one without (π_{THOM}). π_{LAG}, π_{THOM}, D and Q are as described in Eqs.(55 - 58). In these cases, we have $f = 1$, $T = 4$, $\beta = 1$.

Below are the results of varying f.

f	π_{LAG}	π_{THOM}	D	Q
0	43.50	43.50	0	0
0.1	44.08	43.50	0.58	1.33%
0.2	44.76	43.50	1.26	2.90%
0.3	45.54	43.50	2.04	4.68%
0.4	46.40	43.50	2.90	6.67%
0.5	47.35	43.50	3.85	8.86%
0.6	48.40	43.50	4.90	11.25%
0.7	49.53	43.50	6.03	13.86%
0.8	50.76	43.50	7.26	16.69%
0.9	52.09	43.50	8.59	19.74%
1.0	53.53	43.50	10.03	23.05%

Figure 4: Solutions for variation in f. As in figure 3, but here, we solve the models for varying values of f. Here, the fixed parameters are $\alpha = 10$, $T = 4$, $\beta = 1$.

Finally, below, we have results of varying T.

T	π_{LAG}	π_{THOM}	D	Q
1	16	16	0	0
2	34.57	28.25	6.32	22.38%
3	46.13	37.25	8.88	23.85%
4	53.53	43.50	10.03	23.05%
5	58.23	47.50	10.73	22.58%
6	60.99	49.75	11.24	22.60%
7	62.38	50.75	11.63	22.92%
8	62.85	51	11.85	23.23%
9	62.88	51	11.88	23.30%

Figure 5: Solutions for variation in T. As in figures 3 and 4, but here we solve the models for varying values of T. Here, the fixed parameters are $\alpha = 10$, $f = 1$, $\beta = 1$.

5.5 Discussion

In this section we will interpret the results from the previous section.

5.5.1 Initial observations

Initially, we can see that, for our test instances, the profit for the lag model, is never smaller than the profit for the Thomas model, (Thomas 1970), that is, $\pi_{LAG} \geq \pi_{THOM}$. It is currently unknown whether this is true for all possible data sets and functions, but this may be worth investigating in further work.

This observation has a deeper implication than what is obvious. It indicates that the lag effect may be used effectively as a tool, in order to extract more profit from a market.

Furthermore, $\pi_{LAG} = \pi_{THOM}$ for a few instances in particular. Let us take a closer look at what causes this.

Looking at the different values of α, we see that $\pi_{LAG} = \pi_{THOM} = 0$ at $\alpha = 1$ and $\alpha = 2$, but not otherwise. This is due to the cost $h_{1t} \geq 2 \quad \forall\, t = 1, ..., T$. This cost, h_{1t}, is unavoidable, as we are looking at the production horizon problem, and there can only be production in period 1. The highest feasible price in any period is $p_{max} = \frac{\alpha}{\beta} = 2$, and any non-zero demand would require the price to fall. The price would then dip below 2, making production unprofitable. This means that the optimal solution is not to produce at all, earning 0 profit.

For the factor f, the willingness to stock up, $\pi_{LAG} = \pi_{THOM}$ when $f = 0$. This is not surprising, as we recall that we defined our lag function in Eq.(50) as

$$l_t = f\alpha_{t+1} - f\beta_{t+1}p_t \quad \forall\, t = 1, ..., T-1. \tag{64}$$

Observe that f is present in both terms of the lag function. Setting $f = 0$ gives $l_t = 0 \,\forall\, t = 0, ..., T$, removing the only thing that makes the two models different. The optimal solutions will then obviously be the same.

We see a similar effect when setting the amount of time periods, $T = 1$. With only one time period, there cannot be any lag between periods, and in this case the two models are identical.

5.5.2 Variation in α

For α, we see that the difference in optimal objective function values of the two models, D, grows in size as α increases. It starts out at 0, and grows more rapidly as α increases, as we can see in figure 6. Q, on the other hand, starts out high and decreases with increasing α, but at a slower and slower rate.

This indicates that the lag effect may have a greater impact and be more important for high-volume products than for low-volume ones.

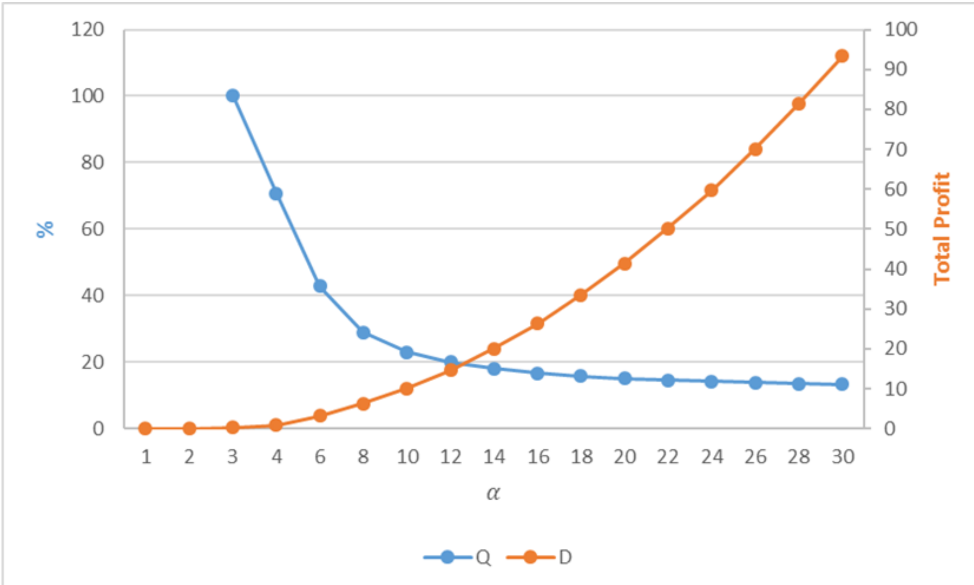

Figure 6: Variation in α. These graphs show the development of D and Q as the value of α increases, see the table in figure 3. Note that Q is measured in percent, on the left vertical axis, while D has no unit and measured on the right vertical axis. In these instances, $f = 1$, $T = 4$, $\beta = 1$.

5.5.3 Variation in f

As we can see from the tables in figures 3 - 5 and the graphs in figure 7, both the absolute difference D and percentage difference Q are zero when $f = 0$. As f increases, they both increase at higher and higher rates.

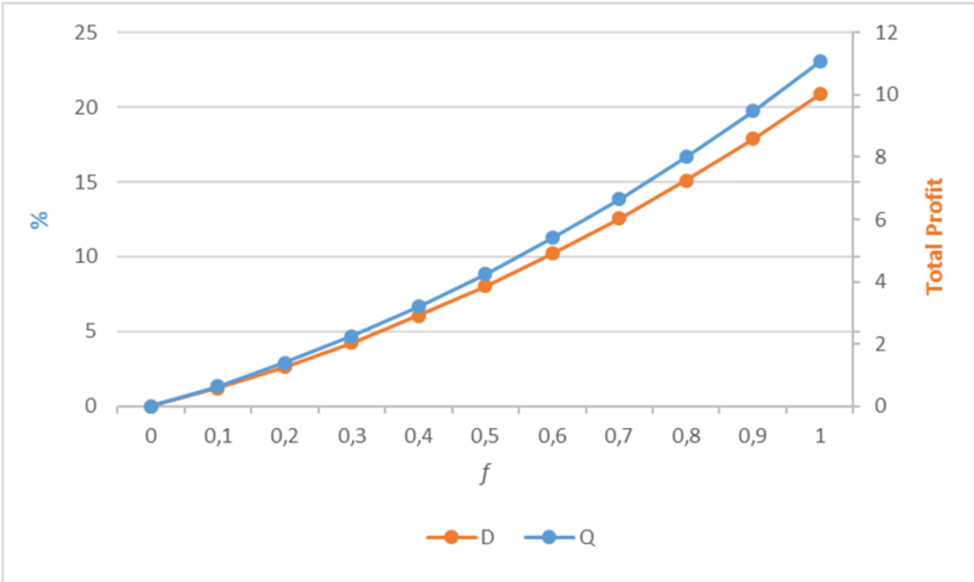

Figure 7: Variation in *f*. As in figure 6, but with varying values of f. This figure plot the values in the table in figure 4. In these instances, $\alpha = 10$, $T = 4$, $\beta = 1$.

In practice, this means that, if the producer can influence customers such that they are more likely to utilize the stockup option, this may be used to increase total profit.

We see from figure 7 that D and Q follow each other closely. This behaviour of both graphs rising at an accelerating pace becomes obvious when we recall that f has no effect on the Thomas model (Thomas 1970). It is only included in the lag function, which only appears in the lag model. This means that π_{THOM} does not change as f increases, while π_{LAG} *does* change, as its demand function becomes increasingly different from that of the Thomas model.

5.5.4 Variation in T

From figure 8, we see that D is non-decreasing over the whole test interval for T, with a steep rise at low values of T. The incline becomes smaller as T increases, flattening out almost completely between $T = 8$ and $T = 9$. Looking at the corresponding table in figure 5, we see not only that the difference between the two models, D, stagnates, but also that the added profit of another time period approaches zero. This is due to the cost h_{1t} approaching the maximum price, $p_{max} = \frac{\alpha}{\beta} = 10$, making it impossible to extract much profit out of the later periods.

This effect of D having a near-zero increase when T is large may be possible to use, in further work, in a heuristic for the full problem. When solving large production horizons, it may be useful to approximate the later periods with the solution for the Thomas model (Thomas 1970), which is requires less computational effort.

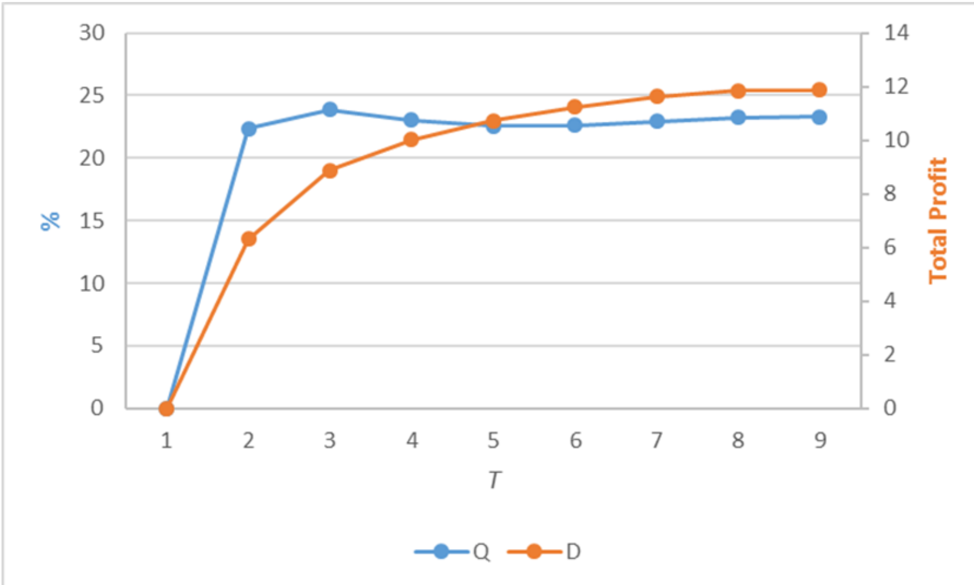

Figure 8: Variation in T. As in figures 6 and 7, but with varying values of T. This figure plot the values in the table in figure 5. In these instances, $\alpha = 10$, $f = 4$, $\beta = 1$.

Also interesting is the development of Q as T increases. From being 0 at $T = 1$, it has a sharp jump to $Q = 22.38\%$, staying between 20% and 25% for the rest of the graph.

6 Conclusion

In this master thesis, we have modelled the profit-maximizing lot size problem with pricing lag effects. We have proved three theorems which restrict the amount of possible optimal solutions, and we have provided a solution to the production horizon problem, a sub-problem of the main problem. We have also run some numerical experiments on the production horizon problem and gained some insight in how the model behaves compared to the same model with no lag effect.

The main results of the experiments are as follows:

- The lag effect has a non-negligible impact on the total profit
- The profit of the lag model seems never to be smaller than that of the same model without the lag effect
- The lag effect seems to have a greater impact on total profit when the sales volume is high
- The lag effect seems to have greater impact when more customers are willing to stock up on the product
- Longer production horizons yield smaller and smaller additional profit for an additional time period

These results indicate that the lag effect is an extension with potential for extracting more profit from a market.

For further work, one may solve the main model of this thesis heuristically, or consider the multi-product problem with a common capacity constraint. Other potential routes include extending the lag effect to reach across more than one period, or develop a similar model which deals with marketing and the marketing lag effect.

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A AMPL code: The 2-period production horizon

This appendix contains the programmed mathematical model, data file, run file and solution file of the 2-period production horizon problem. The model is solved using a solver for quadratic programming problems (MINOS 5.51), and at the same time using the optimal price formulas from chapter 4.2.3. The solution file shows that the solution found by the solver and our formulas are indeed the same.

```
File: 2_period_horizon.mod


param T; #amount of time periods in horizon

set TIME := 1..T;
#set of time periods, i.e. 1...T, for ease of writing the model



param alpha {TIME} >=0;
#constant in base demand function (component independent of lag effect)

param beta  {TIME} >=0;
#slope with respect to price, in base demand function

param h     {TIME} >=0;
#production and inventory holding cost of
production in period i for demand in period j

param L0    >=0;
#lag constant for period 0

param LT    >=0;
#lag constant for period T



var dt    {t in TIME} >=0;
#demand function; consists of base demand function and lag function

var p      {TIME} >=0;
#sales price of the product

var lt     {0..T} >=0;
#lag function; is set to a constant for periods 0 and T,
 an actual function for period 1
```

```
##### VARIABLES FOR CALCULATING OPTIMAL SOLUTION USING OUR FORMULA

var p_opt  {TIME} ;
#optimal price as calculated by our formula

var lt_opt {0..T} ;
#lag function values for the calculated optimal price

var dt_opt {TIME} ;
#demand function values for the calculated optimal price

var profit_opt ;
#total profit for the calculated optimal price

##### OBJECTIVE FUNCTION

maximize Total_Profit: sum{t in TIME} (p[t] * dt[t] - h[t] * dt[t]);

##### CONSTRAINTS

subject to Demand_Func {t in TIME}: #demand function
dt[t] = alpha[t] - beta[t] * p[t] + lt[t] - lt[t-1];

subject to Lag_Func: #lag function for periods 1...T
lt[1] = alpha[2] - beta[2] * p[1];

subject to Lag_0: #lag "function" (set to a constant) for period 0
lt[0] = L0;

subject to Lag_T: #lag "function" (set to a constant) for period T
lt[T] = LT;

##### CONSTRAINTS FOR CALCULATING OPTIMAL SOLUTION USING OUR FORMULA

subject to P_Opt_1: #our formula for the optimal price in period 1
p_opt[1] = (1/(2*beta[1] + (3/2)*beta[2])) *
(alpha[1] + alpha[2] + h[1]*(beta[1] + beta[2]) - (1/2)* beta[2]*h[2]);

subject to P_Opt_2: #our formula for the optimal price in periods 2...T
p_opt[2] = (1/(4*beta[1] + 3*beta[2])) *
(alpha[1] + alpha[2] + h[1]*(beta[1] + beta[2]) -
(1/2)*beta[2]*h[2]) + (1/2)*h[2];
```

```
subject to Lt_Opt_1: #lag function for our calculated optimal price
lt_opt[1] = alpha[2] - beta[2]*p_opt[1];

subject to Lt_Opt_T:
lt_opt[T] = 0;

subject to Dt_Opt {t in TIME}: #demand function for our calculated optimal price
dt_opt[t] = alpha[t] - beta[t]*p_opt[t] + lt_opt[t] - lt_opt[t-1];

subject to Lt_Opt_0: #lag "function" (set to a constant) for period 0
lt_opt[0] = L0;

subject to Profit_Opt: #total profit for our calculated optimal prices
profit_opt = sum{t in TIME} (p_opt[t] * dt_opt[t] - h[t] * dt_opt[t]);
```

File: 2_period_horizon.dat

```
param T := 2;

param alpha := 1 10, 2 10;
param beta  := 1 1, 2 1;

param h     := 1 2, 2 3;

param L0 := 0;
param LT := 0;
```

File: 2_period_horizon.run

```
model 2_period_horizon.mod;
data  2_period_horizon.dat;

option solver 'D:\AMPL\amplide.mswin64\minos';

solve;

display Total_Profit, profit_opt > 2_period_horizon.sol;
display dt,           dt_opt     > 2_period_horizon.sol;
display p,            p_opt      > 2_period_horizon.sol;
display lt,           lt_opt     > 2_period_horizon.sol;
```

File: 2_period_horizon.sol

```
Total_Profit = 34.5714
profit_opt   = 34.5714

:     dt      dt_opt     :=
1   7.14286   7.14286
2   1.71429   1.71429
;

:      p       p_opt     :=
1   6.42857   6.42857
2   4.71429   4.71429
;

:     lt      lt_opt     :=
0   0         0
1   3.57143   3.57143
2   0         0
;
```