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Recoloring subgraphs of K_{2n} for sports scheduling

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ABSTRACT

The exploration of one-factorizations of complete graphs is the foundation of some classical sports scheduling problems. One has to traverse the landscape of such one-factorizations by moving from one of those to a so-called neighbor one-factorization. This approach amounts to modifying locally the coloring associated with a one-factorization. We consider some particular types of modifications and describe various constructions which give one-factorizations which may be modified or not by these techniques. Among those are recoloring of bichromatic cycles, altering of optimally colored subcliques of even size, or recoloring of chordless lanterns.

1. Introduction

A classical model used to solve some sports tournament scheduling problems is the one-factorization of a complete graph K_{2n} on $2n$ nodes. In that model nodes of the graph represent teams, edges represent games to be scheduled, and factors (or colors) represent rounds [10]. A schedule for the tournament is then represented by a one-factorization of K_{2n}. A popular computational approach consists of using local search methods that move from a schedule to a so-called neighbor schedule until getting a locally optimal solution regarding some objective function.

The efficiency of local search relies heavily on the choice of a suitable neighborhood in the set of one-factorizations of K_{2n}. A neighbor solution is found by recoloring an appropriate partial subgraph of K_{2n}. We shall consider in this work some possible choices of the subgraph to be recolored which lead to simple and hence practical recoloring techniques. Such an approach may be viewed as the reconfiguration of a one-factorization. We intend to examine the existence of adequate subgraphs to be recolored. This will lead us to exhibit some properties of one-factorizations that have some interest in their own.

This work deals with the existence of certain colored subgraphs in one-factorizations. We show results both on general one-factorizations and on particular types of one-factorizations.

The rest of the work is organized as follows: In Section 2 we define the colored subgraphs we deal with in this work: colorful chordless lanterns and optimally colored cliques of even size. We also describe some particular types of one-factorizations of interest. In Section 3 we investigate the existence of colorful chordless lanterns in one-factorizations of

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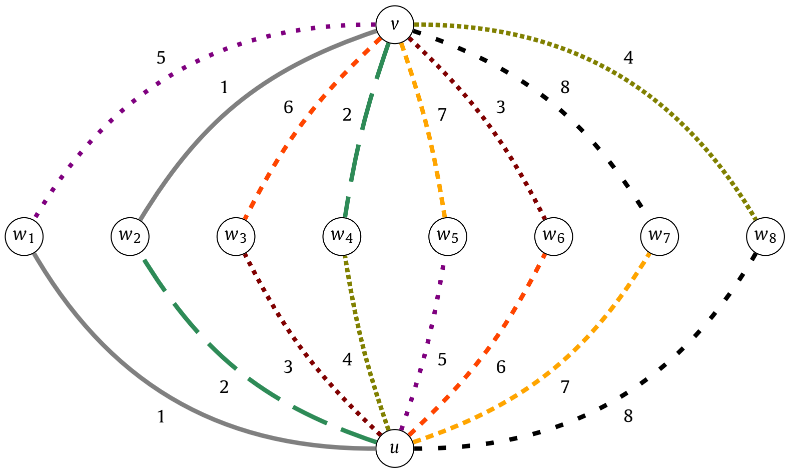

Fig. 1. Two colorful chordless lanterns $L(u, v, W_1)$ and $L(u, v, W_2)$ by taking $W_1 = \{w_1, w_2, w_4, w_5, w_7, w_8\}$ and $W_2 = \{w_3, w_6\}$. It is easy to check that $C(B(u, W_1)) = C(B(v, W_1))$ and $C(B(u, W_2)) = C(B(v, W_2))$.

different sizes and in Section 4 our focus is on the existence of optimally colored cliques of even size. In Section 5 we give results for one-factorizations of graphs with a small number of nodes. The last section consists of concluding remarks.

2. Definitions and basic concepts

In a graph $G = (V, E)$ on node set V and edge set E we consider a node v and a set $W \subset N(v)$ where $N(v)$ is the set of all neighbors of v. Then $B(v, W)$ called the *bundle* of v constructed on W is the set of edges vw with $w \in W$. If $W = N(v)$ we simply write $B(v)$. We will consider complete graphs unless mentioned otherwise. That is $N(v) = V \setminus \{v\}$ for all $v \in V$. For graph theoretical terms and notations not defined in here, the reader is referred to [2].

A *one-factorization* of K_{2n} is a partition of $E(K_{2n})$ into $2n - 1$ *one-factors* $F_1, \dots, F_{2n-1}$. Each one-factor is a perfect matching on K_{2n}. We also call a one-factorization a (proper edge, optimal) coloring and each one-factor is indeed a color class. Since this work does not deal with any other type of factorizations, in the remainder of this manuscript we use the term factorization as an equivalent of the term one-factorization.

It is well-known that a schedule for a single round-robin tournament with $2n$ teams has a one to one correspondence with a (proper edge) coloring of K_{2n} [10]. Neighborhoods used in local search procedures for round-robin tournament scheduling problems can then be associated with partial recolorings of a given coloring. It is interesting then to find, in a given colored graph, subgraphs that can be recolored while maintaining the coloring of the rest of the graph unaltered.

One such type of subgraph is the class of bichromatic cycles. Given a bichromatic cycle, one can exchange the colors of the edges of the cycle, thus obtaining a new proper coloring. If the bichromatic cycle is hamiltonian then the coloring obtained after the recoloring is isomorphic to the original one since the recoloring amounts to just exchanging two one-factors. Whenever the cycle under consideration is not hamiltonian the coloring obtained is different and possibly not isomorphic to the original one.

A factorization $F_1, \dots, F_{2n-1}$ is *perfect* if $F_i \cup F_j$ is a hamiltonian cycle in K_{2n} for any i, j $(i \neq j)$ [8,13]. Notice that in a perfect factorization the recoloring of bichromatic cycles does not allow us to obtain factorizations non-isomorphic to the original one. The size of bichromatic cycles and the perfectness of factorizations have been thoroughly studied. In the context of this work, perfect factorizations are also-called C-blocking (for Cycle blocking) factorizations.

We will study two other types of colored subgraphs that allow local recolorings of factorizations. Those subgraphs are called *colorful chordless lanterns* and *optimally colored even cliques*.

Let v_1, v_2 be two nodes of K_{2n} and $W \subset N(v_1) \cap N(v_2) \setminus \{v_1, v_2\}$. We will consider the subgraph formed by $B(v_1, W) \cup B(v_2, W)$. Let $C(X)$ be the set of colors occurring on the edges of $X \subset E$ in a coloring of K_{2n}. Then, given a coloring of a graph K_{2n} with color set C, if $C(B(v_1, W)) = C(B(v_2, W))$, $W \neq \emptyset$ and inclusionwise minimal for the equality to hold, we say that the subgraph on v_1, v_2, W with edge set $B(v_1, W) \cup B(v_2, W)$ is a (Chinese) *colorful chordless lantern* $L(v_1, v_2, W)$.[1] An illustration is shown in Fig. 1. Here we have two colorful chordless lanterns $L(u, v, W_1)$ and $L(u, v, W_2)$ by taking $W_1 = \{w_1, w_5, w_7, w_8, w_4, w_2\}$ and $W_2 = \{w_3, w_6\}$. A chordless colorful lantern $L(v_1, v_2, W)$ is *trivial* if $W = N(v_1) \setminus \{v_2\} = N(v_2) \setminus \{v_1\}$. In K_{2n}, this means that when there is a trivial colorful chordless lantern, the smallest set W that can be used to construct a colorful chordless lantern has size $2n - 2$.

[1] In [12] a lantern was defined as a graph $K_{2,n-2}$ with bipartition $(\{v_1, v_2\}, V \setminus \{v_1, v_2\})$ plus the edge $[v_1, v_2]$. Here, chordless colorful graphs do not have the edge $[v_1, v_2]$, have a color assigned to each of its edges and obey restrictions on those colors.

12A3948576B 21B3A495867 3124A59687B
4132B5A6978 514236A798B 615243B7A89
71625348A9B 81726354B9A 918273645AB
A192837465B B1A29384756

Fig. 2. An L-blocking one-factorization of K_{12}. Each block of duodecimal digits is a one-factor, with 0 omitted, so that, for example, 12A3948576B denotes the one-factor with edges [01], [2A], [39], [48], [57] and [6B].

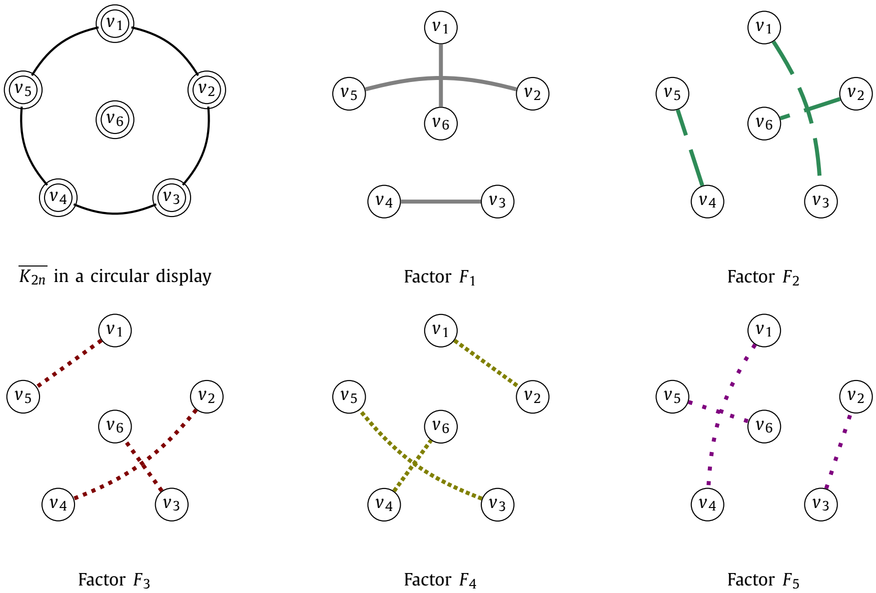

Fig. 3. Canonical factorization of K_6.

Note that it is essential to require that W is inclusion-wise minimal, since otherwise by taking $W = N(v_1) \setminus \{v_2\} = N(v_2) \setminus \{v_1\} = V \setminus \{v_1, v_2\}$ we would always get a trivial colorful chordless lantern.

Furthermore we shall say that a factorization is *L-blocking* (for Lantern blocking) if for any two nodes v_1, v_2 of K_{2n} the colorful chordless lantern $L(v_1, v_2, W)$ is trivial (see Fig. 2). Notice that in an L-blocking factorization, recoloring a colorful chordless lantern $L(v_1, v_2, W)$ does not allow us to obtain a factorization not isomorphic to the original one since the recoloring amounts to exchange the labels of v_1 and v_2.

A factorization is *L-flexible*, if for any two nodes v_1, v_2 of K_{2n} there is a nontrivial colorful chordless lantern $L(v_1, v_2, W)$, i.e. if $|W| < 2n-2$ for any minimal W.

The recoloring operations concerning bi-chromatic cycles and colorful chordless lanterns have been previously studied in [11] under different names.

Let $Y \subset V$ be a set with even cardinality smaller than $2n$. We say that the subgraph induced by Y ($K(Y)$) is an *optimally colored even clique* if $K(Y)$ is colored with $|Y|-1$ colors. We say that a factorization is *K-blocking* (for "Klique" blocking) when such a set does not exist, i.e., the only optimally colored even clique in the graph is the graph itself. Note that in terms of one-factorizations an optimally colored even clique induce a sub one-factorization. With that name optimally colored even clique were previously studied in the literature (see [16]).

We define now some types of factorizations. Most of the results of this work will hold for specific types of factorizations.

We remind that a factorization $F_1, \ldots, F_{2n-1}$ of K_{2n} is called *canonical* whenever $F_i = \{[2n, i] \cup \{[i+k, i-k] | k = 1, \ldots, n-1\}$ for $i = 1, \ldots, 2n-1$ where all integers $i+k, i-k$ are taken modulo $2n-1$ between 1 and $2n-1$. Fig. 3 shows a canonical factorization for K_6.[2]

Assume that the number $2n$ of nodes is divisible by 4 and let us split the set of nodes into two sets $V^1 = \{v_1^1, \ldots, v_n^1\}$ and $V^2 = \{v_1^2, \ldots, v_n^2\}$. We call a factorization of K_{2n} *binary* if it contains a factorization of $K_n(V^1)$ and a factorization of $K_n(V^2)$ using colors $1, \ldots, n-1$. Consequently, colors $n, \ldots, 2n-1$ form a factorization of $K_{n,n}(V^1, V^2)$. Also, if the factorizations of $K_n(V^1)$ and $K_n(V^2)$ are canonical, we call the factorization *bicanonical*.[3]

A binary factorization is *bisymmetric* if it satisfies the following:

[2] The name *canonical* is widely used in sport scheduling literature (see [5,15]). In literature related to one-factorizations of graphs the same factorization is often called a GK_{2n} (see [16]).

[3] A particular bicanonical factorization using a *standard* factorization of $K_{n,n}(V^1, V^2)$ is called GA_{2n} in the lireature (see [16]).

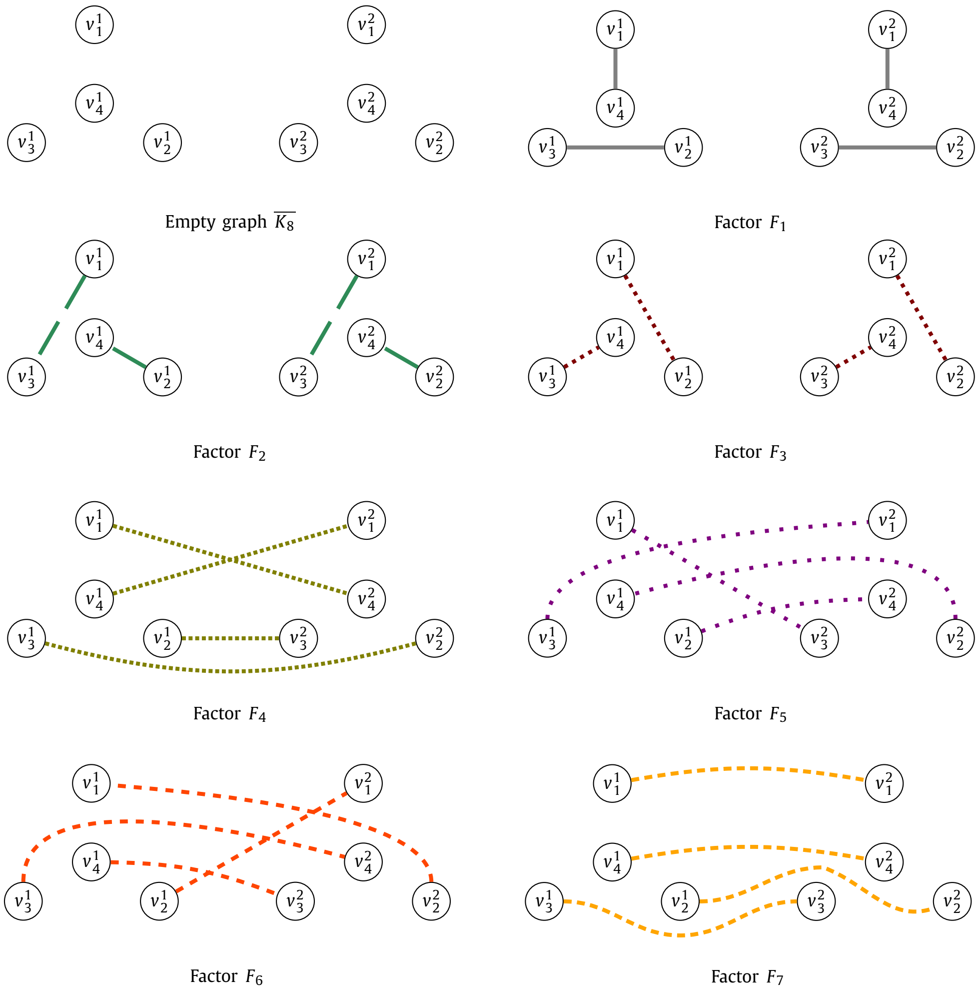

Fig. 4. A bisymmetric one-factorization of K_8.

1. the factorization of $K_n(V^1)$ and that of $K_n(V^2)$ are the same, i.e., $[v_i^1, v_j^1]$ and $[v_i^2, v_j^2]$ have the same color for any $1 \le i < j \le n$;
2. if $[v_i^1, v_j^1]$ has color c, then both $[v_i^1 v_j^2]$ and $[v_i^2, v_j^1]$ have color $c + n - 1$ for any $1 \le i < j \le n$.
3. $[v_i^1, v_i^2]$ has color $2n - 1$ for $1 \le i \le n$

A bisymmetric factorization of K_8 is given in Fig. 4.

The above definitions assume that $2n$ is a multiple of 4. When $2n = 4s + 2$, V^1 and V^2 have $2s + 1$ nodes. One cannot construct a binary factorization as above.

We can however construct a $(2s + 1)$-coloring of $K(V^1)$ and $K(V^2)$ with colors $1, 2, \ldots 2s + 1$. At every node v_j^i of $K(V^i)$ some color among $1, 2, \ldots 2s + 1$ is missing on the edges of $B(v_j^i)$. Since we may take the same coloring for $K(V^1)$ and $K(V^2)$, we may assume that j is the color missing both at v_j^1 and v_j^2. So, we color edges $[v_j^1, v_j^2]$ with color j for $j = 1, 2, \ldots 2s + 1$.

So far we have obtained a factorization $F_1, F_2, \ldots, F_{2s+1}$ of $K(V^1) + K(V^2) + M$ where M is the matching $\{[v_i^1, v_i^2]\, i = 1, 2, \ldots 2s + 1\}$. The edges of $K(V^1, V^2) - M$ (regular bipartite subgraph) can then be colored with $2s$ colors, which gives $F_{2s+2}, \ldots, F_{4s+1}$ (take for instance $F_{2s+p+1} = \{[v_i^1, v_{i+p}^2]\, i = 1, 2, \ldots 2s + 1\}$ for $p = 1, \ldots 2s$ and $i + p$ taken modulo $2s$ between 1 and $2s$).

Such a factorization of K_{4s+2} will be called *almost binary*.

Observation. [6] We recall that there is a simple property which may be used to show that a factorization is not canonical: In a canonical factorization $F_1, \ldots F_{2n-1}$ of K_{2n}, for any choice of $F_i, F_j, F_k (1 \le i < j < k \le 2n-1)$, $G(F_i \cup F_j \cup F_k)$ contains a triangle. This shows, in particular, that binary and almost binary factorizations of K_{2n} with $n \ge 4$ are not canonical.

3. L-blockingness

In this section, we investigate the existence of colorful chordless lanterns of different sizes on different classes of factorizations.

Lemma 1 is Theorem 1.4 in [12].

Lemma 1. *Let G be a graph $K_{2,n-2}$ with bipartition $(\{v_1, v_2\}, V \setminus \{v_1, v_2\})$ plus the edge $[v_1, v_2]$ where $n \ge 4$. Then any edge coloring of G using $2n-1$ colors can be extended to a proper edge coloring of K_{2n} using the same set of colors.*

The first proposition shows that colorful chordless lanterns of any size (with $2 \le |W| \le 2n-2$) may exist in factorizations of K_{2n} for any $n \ge 2$.

Proposition 1. *Any colorful chordless lantern can be extended to a proper coloring of K_{2n}.*

Proof. This is a direct consequence of Lemma 1. We may construct a colorful chordless lantern of any size and then extend the coloring to a graph G as defined in the lemma. The application of the lemma now shows that there is a factorization of K_{2n} containing the constructed colorful chordless lantern. □

In Theorem 2 of [11] the L-blockingness property of factorizations is characterized by the hamiltonian property of the rows (or columns) of the *folds* of the latin square associated with the factorization.

The next proposition characterizes the values of $2n$ for which the canonical factorization of K_{2n} is L-blocking. It uses the concept of faro shuffle permutation. A faro shuffle, also known as riffle shuffle [1], is a permutation π of $2n$ elements from 0 to $2n-1$, such that the sequence $(\pi(0), \ldots, \pi(2n-1))$ is composed by precisely two interleaved increasing sequences. For instance, the faro shuffle permutation of the ordered set $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$ can be expressed as:

$\pi(i)$	0	5	1	6	2	7	3	8	4	9
i	0	1	2	3	4	5	6	7	8	9

To apply a faro shuffle permutation in the ordered set $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$, first split the set in two halves $(0, 1, 2, 3, 4)$ and $(5, 6, 7, 8, 9)$. Then, interleave elements one-by-one from each half to get a new ordering $(0, 5, 1, 6, 2, 7, 3, 8, 4, 9)$. Notice that the elements of each subset $\{3, 6\}$ and $\{1, 2, 4, 5, 7, 8\}$ change places in a cyclic way within its own subset. These permutation subsets are called "orbits" [9]. A list of the values of $2n$ for which the faro shuffle permutes all except the first and last elements, i.e., has an orbit of size $2n-2$, can be found in [14].

Proposition 2. *The canonical factorization of K_{2n} is L-blocking if and only if $2n-1$ is prime and the faro shuffle permutation with $2n$ elements has an orbit of size $2n-2$.*

Proof. This is theorem 1 in [9]. □

Proposition 3. *In the canonical factorization of K_{2n} $(n \ge 2)$, the colorful chordless lantern $L(1, 2n-2, W)$ is trivial.*

Proof. In the canonical coloring of K_{2n} the neighbors of node $2n-2$ in consecutive one-factors $F_1, \ldots F_{2n-1}$ are $3, 5, 7, \ldots, 2n-1, 2, 4, \ldots, 2n-4, 2n, 1$. Those of node 1 are $2n, 3, 5, \ldots, 2n-3, 2n-1, 2, 4, \ldots, 2n-6, 2n-4, 2n-2$. Let $W = V \setminus \{1, 2n-2\}$.

Then, ignoring edge $[1, 2n-2]$ which has color $2n-1$, one can construct a colorful chordless lantern $L(1, 2n-2, W)$. The colors of the edges of each path $(2n-2, w, 1)$ are cyclically consecutive from 1 to $2n-2$ if we consider each possible value of w in the order $3, 5, 7, \ldots, 2n-1, 2, 4, \ldots, 2n-4, 2n$, see Fig. 5. In consequence it is not possible to take any proper subset of W to obtain a nontrivial colorful chordless lantern. □

The previous proposition shows how to find a trivial colorful chordless lantern in the canonical coloring of K_{2n}.

Corollary 1. *The canonical factorization of K_{2n} is not L-flexible for any $n \ge 2$.*

The next proposition shows that the presence of trivial colorful chordless lanterns is not restricted to canonical colorings.

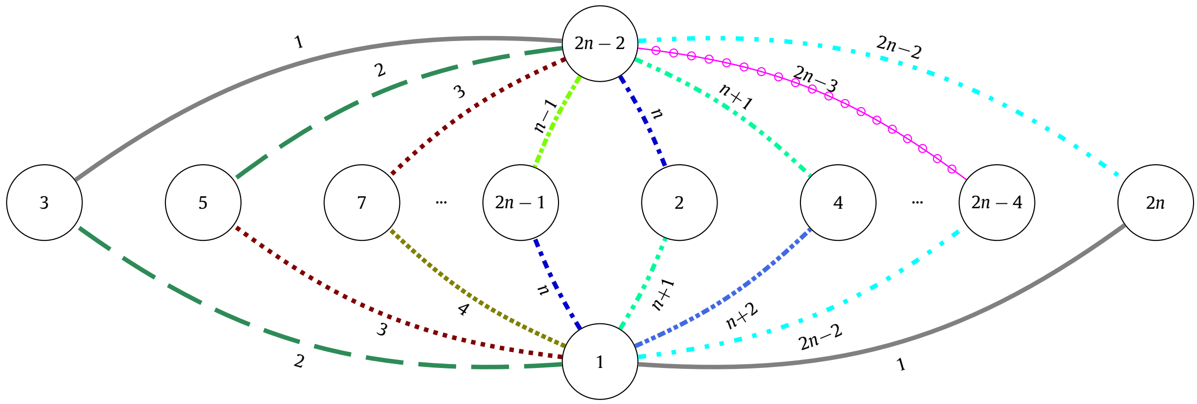

Fig. 5. A trivial colorful chordless lantern $L(1, 2n-2, W)$.

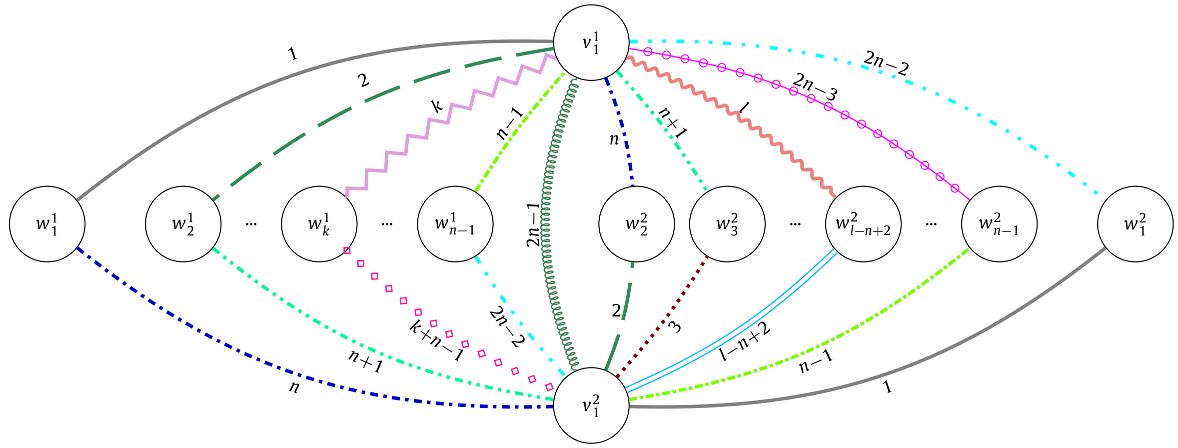

Fig. 6. Initial pre-coloring.

Proposition 4. *Whenever n is even there is a binary factorization of K_{2n} that is not L-flexible.*

Proof. Nodes v_1^i, w_1^i, …, w_{n-1}^i are the nodes of V^i $(i = 1, 2)$. First, we construct a trivial lantern $L(v_1^1, v_1^2, W)$ as shown in Fig. 6.

For $k = 1$ to $n-1$, we have edges $[v_1^1, w_k^1]$ with color k and $[w_k^1, v_1^2]$ with color $k+n-1$.

For $l = n$ to $2n-3$ we have edges $[v_1^1, w_{l-n+2}^2]$ with color l and $[w_{l-n+2}^2, v_1^2]$ with color $l-n+2$ and finally we have $[v_1^1, w_1^2]$ with color $2n-2$ and $[w_1^2, v_1^2]$ with color 1.

So edges $[v_1^1, w_k^1]$ for $k = 1, \dots, n-1$ are in $K(V^1)$ and they are the edges adjacent to v_1^1 with colors $1, \dots, n-1$ in $F_1, \dots, F_{n-1}$.

Similarly, edges $[v_1^2, w_{l-n+2}^2]$ for $l = n, \dots, 2n-3$ are in $K(V^2)$, they are the edges adjacent to v_1^2 with colors 2, …, $n-1$ in F_1, …, F_{n-1}.

Furthermore, $[v_1^2, w_1^2]$ is the edge of color 1 adjacent to v_1^2 in $K(V^2)$.

One verifies that $L(v_1^1, v_1^2, W)$ constructed above is a trivial colorful chordless lantern: starting with $[v_1^1, w_1^1]$ of color 1, we go through $[w_1^1, v_1^2]$ of color n, then from $[v_1^1, w_2^2]$ of color n, we go through $[w_2^2, v_1^2]$ of color 2 and we continue. The colors of the edges followed are consecutively $1 \to n \to 2 \to n+1 \to \dots \to k \to k+n-1 \to \dots \to 2n-3 \to n-1 \to 2n-2 \to 1$.

Now the edges of color $n, n+1, \dots, 2n-2$ in $L(v_1^1, v_1^2, W)$ together with edge $[v_1^1, v_1^2]$ that is colored with color $2n-1$, give us a set of precolored edges in $K(V^1, V^2)$. See Fig. 7.

We now construct the factorization. Renaming and reordering the nodes w_i^1, w_i^2 $(i = 1, \dots, n-1)$ our problem now reduces to extending a precoloring of a complete bipartite graph $K(\overline{V^1}, \overline{V^2})$ with $\overline{V^i} = \{v_1^i, \dots, v_n^i\}$, $(i = 1, 2)$, edges $[v_1^1, v_j^2][v_j^1, v_1^2]$ precolored with color $n-2+j$ $(j = 2, \dots, n)$. We can take the last one-factor in the coloring as $F_{2n-1} = [v_1^1, v_1^2] \cup \{[w_j^1, w_j^2] \mid j = 2, \dots, n\}$.

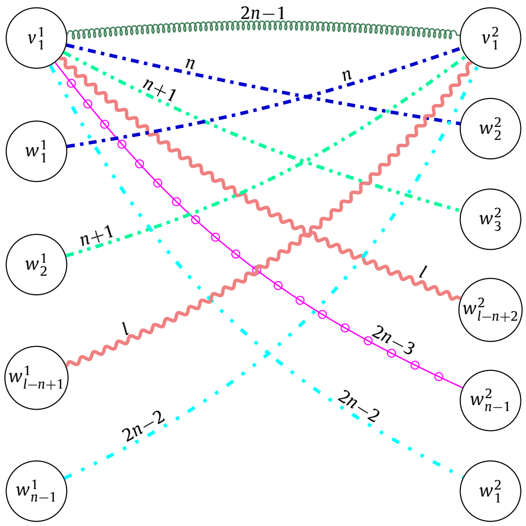

Fig. 7. Precoloring.

Consider now an arbitrary one-factorization $\mathcal{F} = (\hat{F}_1, \ldots, \hat{F}_{n-1})$ of K_n. We construct F_{n-2+j} with precolored edges $[v_1^1, v_j^2], [v_j^1, v_1^2]$ as follows:

Let $\hat{F}_k$ be the one-factor of $\mathcal{F}$ containing edge $[1, j]$; we consider every edge $[p, q]$ of $\hat{F}_k$ ($\neq [1, j]$) and introduce $[v_p^1, v_q^2]$ and $[v_q^1, v_p^2]$ into F_{n-2+j}.

We repeat this construction for $j = 2, \ldots, n$.

It will give us the one-factors $F_n, F_{n+1}, \ldots, F_{2n-2}$ which together with $F_1, \ldots, F_{n-1}$ and F_{2n-1}, obtained earlier, will give us the required factorization of K_{2n}. □

The following proposition shows that bisymmetric factorizations are L-flexible.

Proposition 5. *In a bisymmetric factorization of K_{2n} for any two nodes u, v there is a colorful chordless lantern $L(u, v, W)$ with $|W| = 2$.*

Proof. The nodes are divided in two sets $V^1 = \{v_1^1, \ldots, v_n^1\}$ and $V^2 = \{v_1^2, \ldots, v_n^2\}$. We have three cases to examine:

1. $\underline{u, v \in V^1 \text{ or } u, v \in V^2}$: Without loss of generality, let $u = v_1^1$ and $v = v_2^1$. Consider $[v_1^1, v_1^2]$; its color is $2n-1$. If $[v_1^1, v_2^1]$ has color c, edge $[v_1^2, v_2^1]$ has color $c + n - 1$; by construction $[v_1^1, v_2^2]$ has the same color and $[v_2^1, v_2^2]$ has color $2n-1$. These edges form a colorful chordless lantern $L(v_1^1, v_2^1, W)$ with $W = \{v_1^2, v_2^2\}$ and colors $c + n - 1$ and $2n - 1$.
2. $\underline{u \in V^1, v \in V^2}$: Let, without loss of generality, $u = v_1^1$
   (a) $\underline{\text{assume } v = v_i^2, i \neq 1}$; then $[v_1^1, v_i^1]$ has some color c, $[v_1^1, v_1^2]$ and $[v_i^1, v_i^2]$ have color $2n-1$; $[v_1^2, v_i^2]$ also has color c and we have a colorful chordless lantern $L(v_1^1, v_i^2, W)$ with $W = \{v_i^1, v_1^2\}$ and colors c and $2n-1$.
   (b) $\underline{\text{let } v = v_1^2}$; then consider edge $[v_1^1, v_2^1]$; it has the same color c, as does edge $[v_1^2, v_2^2]$; the edges $[v_1^1, v_2^2]$ and $[v_1^2, v_2^1]$ have color $c + n - 1$. They form a colorful chordless lantern $L(v_1^1, v_1^2, W)$ with $W = \{v_2^1, v_2^2\}$ and colors c and $c + n - 1$. □

The following easy proposition shows that binary factorizations are not L-blocking.

Proposition 6. *A binary factorization of K_{2n} has a colorful chordless lantern $L(u, v, W)$ with $|W| \leq n - 2$.*

Proof. Take any two nodes u, v, both belonging to $K_n(V^1)$. Call W the set of the other nodes in the subgraph. Observe that the set of colors of edges $[u, w], w \in W$ are the same as the set of colors of edges $[w, v], w \in W$ and, in consequence, u, v and W induce a colorful chordless lantern or there is a $W' \subset W$ for which u, v and W' induce a colorful chordless lantern. The set W has size at most $n - 2$. □

We have shown that canonical factorizations of K_{2n} are L-blocking for certain values of $2n$ and that binary factorizations are not L-blocking. Then, it is natural to ask if there is a non-canonical L-blocking factorization of K_{2n} for any n. In [11] it is shown that K_{12} has 8 L-blocking factorizations (7 of them not canonical).

Some of the above propositions show that certain factorizations including the canonical and some binary are not L-flexible. Then, Proposition 5 shows that bisymmetric factorizations are L-flexible. One would like to know if L-flexible factorizations exist for values of $2n$ with n odd. The cases with $2n \leq 10$ are investigated in section 5, but for larger values of $2n$ we state the following open problem.

Open problem 1. *Is there at least one L-flexible factorization of K_{4s+2} for each $s \geq 3$?*

4. Optimally colored even cliques of one-factorizations

In this section, we study optimally colored even cliques inside factorizations. We notice that any factorization of K_{2n} is an optimally colored even clique of size $2n$. So, here we deal with proper subgraphs of K_{2n} inducing optimally colored even cliques.

First we note that optimally colored even cliques of any size r ($4 \leq r \leq n$) exist in some factorizations of K_{2n}.

Observation. For any two even numbers r and n larger than 2, a factorization of K_r can be extended to a factorization of K_{2n} if and only if $r \leq n$

For a proof, see Theorem 2 in [4] or Theorem 14.2 in [16].

We continue with a complete characterization of the optimally colored cliques within the canonical factorizations of K_{2n}.

Proposition 7. *A canonical factorization of K_{2n} contains an optimally colored even clique K_{2p} ($p \leq n$) if and only if $2n = (2p-1)s+1$.*

Proof. Notice that s is necessarily odd. To have a canonical coloring we place nodes $1, 2, \dots, 2n-1$ on a circle with the same distance between any consecutive nodes and node $2n$ is in the center of the circle as in Fig. 3.

a) Assume first that $2n = (2p-1)s+1$ for some fixed p and s. Consider the set $W = \{1, s+1, 2s+1, \dots, (2p-2)s+1\}$; there are exactly $s-1$ nodes between any two consecutive nodes of W ($(2p-2)s+1$ and 1 are considered consecutive in W). Let F'_i be the edges of F_i with both endpoints in $W \cup \{2n\}$ for $i = 1, s+1, 2s+1, \dots, (2p-2)s+1$. We have for instance $F'_1 = \{[2n, 1], [s+1, (2p-2)s+1], [2s+1, (2p-3)s+1], \dots [ps+1, (p+1)s+1]\}$. Since $s-1$ is even all these p edges are indeed in F_1 and it is a perfect matching in $W \cup \{2n\}$. This last property holds for all $F_i, i = 1, s+1, 2s+1, \dots, (2p-2)s+1$. So, $F'_1, F'_{s+1}, F'_{2s+1} \dots, F'_{(2p-2)s+1}$ is a factorization of the complete subgraph of K_{2n} induced by $W \cup \{2n\}$ which has $2p$ nodes.

b) Conversely, let us assume that there is no s such that $2n = (2p-1)s+1$. In other words $2n-1 \neq (2p-1)s$ for any integral (odd) s.

First, we show that such a K_{2p} must necessarily contain node $2n$. Consider a canonical coloring of K_{2n} and let $\overline{F}_1, \dots, \overline{F}_{2n-1}$ be the coloring induced on $K_{2n} - 2n$. Each $\overline{F}_i$ consists of parallel edges in Fig. 3.

Claim 1. *There is no optimally colored clique K_{2p} in the canonical factorization of K_{2n} without node $2n$ for $p \geq 2$.*

Proof. Let W be a set of $2p$ nodes of K_{2p} placed in the circle among $1, 2, \dots, 2n-1$. If $K(W)$ is optimally colored, the canonical coloring of K_{2n} induces on $K(W)$ a coloring $\hat{F}_{a_1}, \dots, \hat{F}_{a_{2p-1}}$ with colors $a_1, a_2, \dots, a_{2p-1} \subseteq \{1, ..., 2n-1\}$ and each $\hat{F}_i$ consists of parallel edges since $\hat{F}_i \subset \overline{F}_i$. But it is not possible to find such a coloring where each $\hat{F}_i$ consists of parallel edges covering exactly the nodes of W: if the nodes of W, in the order they appear in the circle, are $b_1, b_2, \dots, b_{2p-1}$, the perfect matching $\hat{F}_g$ containing $[b_1, b_3]$ cannot consist of parallel edges since the edge $[b_2, b_l] \in \hat{F}_g$ will cross with $[b_1, b_3]$ for any $4 \leq l \leq 2p-1$. Therefore, no such optimally colored K_{2p} can possibly exist. □

So, an optimally colored K_{2p} involves node $2n$ and $2p-1$ nodes $a_1, a_2, \dots, a_{2p-1}$ distributed arbitrarily among positions $1, 2, \dots, 2n-1$ on the circle. Since $2n \neq (2p-1)s+1$ there will necessarily be three (cyclically) consecutive nodes in the set $\{a_1, a_2, \dots, a_{2p-1}\}$, say a_{2p-1}, a_1, a_2, such that the number d of nodes of the circle between a_{2p-1} and a_1 is different from the number c of nodes between a_1 and a_2. Now, since we have a canonical coloring the matching containing $[2n, a_1]$ should contain $[a_{2p-1}, a_2]$, but this is impossible since d is different from c. So, we cannot find an optimally colored K_{2p}. □

In the previous proposition, if $s = 1$, then $p = n$ and the result is trivial. So, if we have $s \geq 3$, the largest value of p is obtained for the smallest odd s such that $p = \frac{2n+s-1}{2s}$. We cannot find values of s and p satisfying the equality if $2n-1$ is prime. Then, the next corollary characterizes the values of $2n$ for which the canonical coloring of K_{2n} is K-blocking.

Corollary 2. *The canonical coloring of K_{2n} is K-blocking if and only if $2n-1$ is prime.*

Next, we study binary factorizations. The next remark notes that they are trivially not K-blocking.

Table 1
A classification of the six non-isomorphic factorizations of K_8.

1	1234567 2134657 3124756 4152637 5142736 6172435 7162534	L-flexible
2	1234567 2134657 3124756 4152637 5142736 6172534 7162435	L-flexible
3	1234567 2134657 3124756 4162537 5172634 6142735 7152436	-
4	1234567 2134657 3124756 4162735 5172634 6142537 7152436	-
5	1234567 2134657 3142756 4162537 5172634 6123547 7152436	K-blocking
6	1234567 2143657 3162547 4172635 5123746 6152734 7132456	C-blocking and K-blocking

Observation. Binary factorizations of K_{2n} have, by definition, two optimally colored cliques of size n.

The next proposition shows that for some values of $2n$ there are almost binary factorizations of K_n that are not K-blocking.

Proposition 8. *For K_{4s+2} there is an almost binary factorization which contains two optimally colored cliques of size $s+1$ if s is odd.*

Proof. We divide the $4s+2$ nodes into $V^i = \{v^i_1, ..., v^i_s\}$ and $W^i = \{w^i_1, ..., w^i_{s+1}\}$ for $i = 1, 2$. In this proof, the factorizations constructed for $K(V^i \cup W^i), i = 1, 2$, are symmetrical. For i=1,2 we construct in $V^i \cup W^i$ a factorization of the complete graph $K(W^i)$ with colors $1, 2, ..., s$ and a coloring of $K(V^i)$ with colors $1, 2, ..., s$. These colorings exist since $s+1$ is even (and s is odd). Let j be the color among $1, 2, ..., s$ which is missing on the edges of $B(v^i_j)$. Color edge $[v^1_j, v^2_j]$ with color j for $j = 1, ..., s$.

Then for $i = 1, 2$ we color the edges of the complete bipartite graph $K(W^i, V^i \cup v^i_0)$ (where v^i_0 is an artificial node) with colors $s+1, ..., 2s+1$. Consider the edges $B(v^i_0)$ for $i = 1, 2$: if $[v^i_o, w^i_j]$ has color c $(s+1 \le c \le 2s+1)$ then replace $[v^1_0, w^1_j]$ and $[v^2_0, w^2_j]$ by $[w^1_j, w^2_j]$ and give it color c.

So far we have obtained $2s+1$ one-factors of $K(V^1 \cup W^1) \cup K(V^2 \cup W^2) + \mathcal{M}$ where $\mathcal{M}$ is a matching of size $2s+1$ between $V^1 \cup W^1$ and $V^2 \cup W^2$ whose edges have colors $1, 2, ..., 2s+1$.

The edges of $K(V^1 \cup W^1, V^2 \cup W^2) - \mathcal{M}$ can then be colored with colors $2s+2, ..., 4s+1$ (since the degree of each node is $2s$).

In the factorization of K_{4s+2} constructed, $K(W^1)$ and $K(W^2)$ are optimally colored cliques of size $s+1$. □

Before ending this section the next remark shows that K-blockingness is a precondition for both L-blockingness and perfectness.

Observation. In a non-K-blocking factorization of K_{2n} there is at least one optimally colored even clique $K_r, r \le n$. Taking any 2 nodes of the clique u, v and setting W as the nodes of K_r minus u and v, we obtain a chordless colorful lantern $L(u, v, W)$. If we take the edges of K_r with any of two colors used in K_r, we obtain a cycle or a set of cycles through the nodes of K_r. Then, non-K-blocking factorizations of K_{2n} are neither L-blocking nor perfect.

5. Results for one-factorizations of small complete graphs

In this section, we show some blocking results for factorizations of K_{2n} for small values of $2n$. For $2n \le 10$ we were able to obtain results by enumeration of all non-isomorphic factorizations or K_{2n}.

For $2n = 4$ and $2n = 6$ there is only one non-isomorphic factorization of K_{2n}. Both are K-blocking, perfect, and L-blocking.

For $2n = 8$ there are 6 non-isomorphic factorizations of K_8. Table 1 shows all of the 6 factorizations of K_8 and classifies them according to the blocking properties studied in this work. Notice that there is no L-blocking factorization of K_8.

After checking each factorization of K_{10} available on [3] we obtained the following classification: 227 are just K-blocking, one (the canonical) is C-blocking and K-blocking, three are L-flexible and the remaining 115 do not fall in any category. There is no L-blocking factorization of K_{10}.

In [11] it is shown that K_{12} has 8 L-blocking factorizations.

Observation. There are at least five non-isomorphic K-blocking and one L-flexible factorizations of K_{12}.

Proof. There are 526,915,620 non-isomorphic one-factorizations of K_{12} [7]. Among those factorizations, there are five that are C-blocking [7] and by Observation 4 they must be $K-blocking$. Moreover, by Proposition 5 the (unique on isomorphism) bisymmetric factorization of K_{12} is L-flexible. □

6. Concluding remarks

In this work, we have studied some blocking properties of one-factorizations of complete graphs K_{2n}. For this purpose we have introduced two types of colored subgraphs: colorful chordless lanterns $L(v_1, v_2, W)$ and optimally colored even cliques K_{2p}.

These two classes of subgraphs, together with bichromatic cycles, play an important role in recoloring procedures commonly used in algorithmic approaches for sport scheduling problems. With these new concepts in hand, we classified one-factorizations in terms of the existence or not of non-trivial subgraphs of each class. In L-blocking factorizations, there are no non-trivial colorful chordless lanterns and in K-blocking factorizations there are no non-trivial optimally colored even cliques.

Among other results, we characterized the values of $2n$ for which the canonical factorization of K_{2n} is L-blocking, showed that the canonical factorization is never L-flexible, determined that there are non-canonical non-L-flexible factorizations and showed how to construct a L-flexible factorization whenever n is even.

Concerning K-blocking, among other results, we characterized the values of $2n$ for which the canonical factorization of K_{2n} is K-blocking and proved that there are almost binary factorizations that are not K-blocking.

Observation 4 showed that K-blockingness is a precondition to both perfection, i.e., C-blockingness, and L-Blockingness. Moreover, Corollary 2 shows that the canonical factorization of K_{2n} is K-blocking if and only if $2n - 1$ is prime. These two facts combined with Proposition 2 show that $2n - 1$ has to be prime for the faro shuffle permutation with $2n$ elements to have an orbit of size $2n - 2$. Also, knowing that the canonical factorization is perfect whenever $2n - 1$ is prime [16], we conclude that all L-blocking factorizations found in this work are perfect. This might signal that perfection may be a necessary condition for a factorization to be L-blocking. In [11] it is shown that K_{12} has 8 L-blocking factorizations and only 5 perfect factorizations disproving the claim.

As future work we intend to study prohibited precolorings for L-blockingness and K-Blockingness.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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