Uladzimir Rubasheuski

Models and algorithms for coordinated lot-sizing and joint replenishment

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Preface

This dissertation presents the summary of the research work performed to obtain a PhD degree in Logistics at Molde University College, Molde, Norway.

I have been employed by the college as a Research Fellow for four years and three months starting from October 2011 until January 2016. The research was supervised by Associate Professor Johan Oppen from Molde University College, Norway and Professor David L. Woodruff from University of California, Davis, USA.

The dissertation consists of four papers and an introduction devoted to studies of stochastic joint replenishment and capacitated lot-sizing problem. A general overview related to the joint replenishment and the lot-sizing problems is provided in the introduction part of the dissertation. Problem formulations given in the papers are based on the special case of Stokke AS, and solution algorithms are tested on the data provided by the company. The main attribute of the presented problem formulations, making them different from the existent research, is the piecewise linear structure of transportation costs. To solve the proposed problems author develops enumeration and stochastic programming algorithms. To facilitate the stochastic programing algorithm a new method of scenario generation is developed and presented in paper 2.

The evaluation committee for this work is Professor Maria Grazia Speranza from Universita degli Studi di Brescia, Brescia, Italy, Professor Asgeir Tomasgard from Norwegian University of Science and Technology, Trondheim, Norway, and the local Associate Professor Arild Hoff from Molde University College, Molde, Norway.

Acknowledgments

First and foremost I would like to thank my supervisors Associate Professor Johan Oppen and Professor David L. Woodruff for the help and support they provided me during the years of my PhD research. I have had Johan Oppen as a supervisor of my MSc Thesis, and doubtlessly without his support and belief success of my application to the PhD position was questionable. I am pleased to say that Johan Oppen was not only my supervisor during the years of studies, but become my friend.

I grateful to my co-supervisor Professor David L. Woodruff for his fruitful comments, which helped to clarify and polish the papers. I am thankful for the hospitality he surrounded me with during my stay at University of California in Davis. He managed not only the research process, but also solved most of the organizational issues, allowing me to concentrate on the main purpose of my visit and benefit from all the opportunities University of California can provide.

The whole research in a great extent is the result of my cooperation with Stokke AS, and particularly with Supply Chain director Nils Laugesen. I am thankful to Nils for his interest in my research and willingness to share information about Stokke. He is the one who helped me to collect sufficient information to conduct the research, not the least, by offering me a three-months internship as a Supply Chain coordinator at Stokke AS.

I appreciate the fruitful discussions I had with my colleges at Molde University College, especially with my office mates Jorge Luis Oyola Mendoza, Deodat Edward Mwesiumo, Katerina Shaton and Dušan Hrabec.

I am grateful to my family and parents for their love, understanding and support. I am thankful to my parents Ala Zan and Rubasheuski Yuri for the freedom of choice they gave me. And even though at the age of 6 I preferred theater school rather than math school, my decision was never challenged. I am thankful to my wife Anastasia Rubasheuskaya and my son Aleksander Rubasheuski for the love they surrounded me with, for their ability to release my tension and for their support in the decisions I make.

And last but not least, I like to thank the staff at Molde University College for organizing the study process and solving whatsoever issues.

Molde, Norway
March, 2016

Uladzimir Rubasheuski

Contents

Introduction

Introduction

Inventory management as well as transportation is crucially important for a large number of companies operating in continuously changing markets. Since the introduction of the single item economic order quantity (EOQ) model by F.Harris (1913), the problem of trading off inventory costs against ordering (transportation) costs has been in focus of numerous researchers. Nevertheless, challenges such as development of robust solution methods for the coordinated stochastic lot-sizing problem still stand.

Joint replenishment and coordinated lot-sizing in a stochastic environment gained the attention of the researchers a number of years ago. Sox et al. (1999) presented a review of the literature on the stochastic economic lot scheduling problem (SELSP), considering the stochastic capacitated lot sizing problem as a version of it. Khouja and Goyal (2008) presented the most recent review of the literature on the joint replenishment problem (JRP), including a section on the stochastic JRP. They indicated a can-order policy (Balintfy, 1964) among the first solution approaches developed to deal with the SJRP. Despite the broad variety of literature on the topic, majority of the research simplify the structure of the transportation costs, what complicates the implementation of the developed solution methods on practice.

The focus of this thesis is on the development of solution methods for the stochastic joint replenishment problem and coordinated stochastic lot-sizing problem with a piecewise linear structure of transportation costs. In the following, a brief review of inventory management in a stochastic demand setting is given. After that, the contribution of this research and the summary of the four papers are presented.

1.1 Inventory management in a stochastic demand environment

Inventory management, as well as any other major part of Logistics and Supply Chain Management, is broad and has roots in ancient times. One of the very first documented examples of Inventory Management application on practice was presented in the Bible (Genesis,41). The Egyptians collected harvest for seven productive years to supply the demand for the next poor years. Since those times not much has changed, and inventory management practitioners are dealing with all kind of problems of demand satisfaction using the inventory.

The modern era of inventory management started with the introduction of the EOQ (economic order quantity) formula (Harris, 1913) and an introduction of the general formulation for the newsvendor problem (Arrow et al., 1951). Their combination led to the development of a series of inventory replenishment policies (systems), each of which relies on the function of the total costs, which includes, ordering, holding and backordering costs. Robinson et al. (2009) considers the joint replenishment problem (JRP) and the economic lot scheduling problem (ELSP) as the natural extensions to the EOQ model presented by Harris (1913). According to Sox et al. (1999), the ELSP is a generalized version of the lot-sizing problem. The stochastic joint replenishment problem and the stochastic lot-sizing problem were taken as the starting point for the current research.

1.1.1 The Joint Replenishment Problem

Settings of the JRP include total inventory holding costs and ordering costs, consisting of individual and joint setup costs, which are minimized over an infinite time horizon.

Table 1.1: Notation

Sets	
K	a set of integer multipliers
Parameters	
TC	total annual holding and ordering costs
i	1,2,...*n*, a product index
n	number of products
S	major ordering cost for each replenishment
s_i	minor ordering cost, occurred if product *i* is ordered in a replenishment
h_i	annual holding cost of product *i*
d_i	annual demand for product *i*
$I_{i,0}$	initial inventory of product i
Variables	
T	time interval between successive replenishments
k_i	integer multiplier of replenishment cycle length for each product *i*
$X_{i,t}$	quantity of product i produced in period t
$I_{i,t}$	inventory of product i in the end of period t
$Y_{i,t} \in \{0,\ 1\}$	decision variable indicating whether production of product i takes place in period t ($Y_{i,t} = 1$)

The classical formulation for the JRP are presented below using the notation from Table 1.1:

$$\mathit{min} TC\left(T,\ K\right) = \frac{T}{2}\sum_{i=1}^{n} k_i d_i h_i + \frac{\left(S + \sum_{i=1}^{n} \frac{s_i}{k_1}\right)}{T} \tag{1.1}$$

While solving the JRP problem one aims to find an optimal length of the replenishment cycle interval and a set of integer multipliers. Shifting from the deterministic demand to the stochastic leads to the inclusion of safety stock and backordering (lost sales) costs in the model. Khouja and Goyal (2008) presented the most recent review of the literature on this topic. They indicated the two most common policies to deal with the problem: a periodic review policy and a can-order policy.

One of the first periodic review policies was developed by Atkins and Iyogun (1988). They proposed a $(T,\ M_i)$ policy, where all the products were reviewed after time interval *T* and ordered up to the level M_i. Viswanathan (1997) developed a periodic $(m,\ M)$ policy. Following this policy, all items are review each T time units, and items with inventory lower then m_i level are ordered to M_i level. If to fix the time interval T, than the optimal pair $(m,\ M)$ can be found using the Zheng and Federgruen (1991) algorithm.

Another class of policies can be related to the can-order policy type, which assumes a continuous review of inventory level. Whenever inventory for any of the products drops down to the must-order level, m_i, this product is replenished up to level M_i. All other products $j \neq i$ with inventory level lower than the can-order level, c_j, are also ordered to the level M_j. According to Pantumsinchai (1992), the can-order policy outperforms the periodic review policy for problems with small ordering costs. Melchiors (2002) proposed an improvement to the can-order policy, which was further developed by Johansen and Melchiors (2003). The compensation approach developed by the authors indicates that the major ordering costs assigned to the product considered for replenishment should be discounted by the expected value of the benefit of other products with low-inventory included in the order.

A policy proposed by Ozkaya et al. (2006) combined features of the periodic review and the can-order policies. Following this policy, the inventory level is reviewed continuously and whenever aggregated demand reaches level A, or time since last replenishment reaches T units, the order for all products is placed up to level M_i. Compared to the modified periodic review policy and can-order policy with compensation approach, this policy demonstrated 1,14% cost improvement from the second best policy for 100 out of 139 instances from Atkins and Iyogun (1988) and Viswanathan (1997).

The model discussed in Paper 1, presented later, is based on the periodic review replenishment system, determining the optimal pair of review period length and upper level of inventory. The analyzed problem differs from the previous research on the topic by inclusion of heterogeneous fleet where an unlimited number of vehicles of different sizes are used for transportation, and both full truck and less than full truck loads are allowed. This leads to the replacement of minor ordering costs by direct transportation costs, which are no longer linear and not continuous. The author developed an algorithm capable to solve such problems to optimality for instances of real world size.

1.1.2 The Lot-Sizing Problem

Paper 2 and 3 of this thesis are devoted to the solution of the coordinated capacitated stochastic lot-sizing Problem. The lot-sizing problem (LSP) is another extension of the EOQ problem. As apposed to the solution of the JRP, which is a production (ordering) policy, the solution of the LSP is a production (ordering) plan for a finite time horizon.

The classical formulation for the deterministic LSP is presented below using the notation from Table 1.1:

Table 1.2: Formulation for the deterministic uncapacitated LSP

$$minTC = \sum_{i=1}^{n}\sum_{t=1}^{\tau}(SY_{i,t} + h_i I_{i,t}) \tag{1.2}$$

Subject to:

$$I_{i,t-1} + X_{i,t} = d_{i,t} + I_{i,t},\ \forall\, i = 1,..,n,\ t = 1,...,\tau \tag{1.3}$$

$$X_{i,t} \leq Y_{i,t}\sum_{t=1}^{\tau} d_{i,t},\ \forall\, i = 1,..,n,\ t = 1,...,\tau \tag{1.4}$$

$$X_{i,t} \geq 0,\ \forall\ \forall i = 1,..,n,\ t = 1,...,\tau \tag{1.5}$$

$$I_{i,t} \geq 0,\ \forall\ \forall i = 1,..,n,\ t = 1,...,\tau \tag{1.6}$$

$$Y_{i,t} \in \{0,\, 1\},\ \forall\ \forall i = 1,..,n,\ t = 1,...,\tau \tag{1.7}$$

While solving this variant of the LSP one aims to minimize the sum of the setup and inventory costs, satisfying the demand. Different extensions to the classical LSP are best presented with the use of the taxonomy proposed by Robinson et al. (2009) (Figure 1.1). The authors applied the taxonomy to classify the literature on the problems under deterministic dynamic demand, but it also suits the problems with stochastic demand as well.

The focus of the present thesis is on multi-item problems. The amount of literature on such problems with stochastic demand is limited. Authors have mostly concentrated on the development of models and algorithms for the multi-item capacitated lot-sizing problems (MCLSP). Sox and Muckstadt (1999) were among the first dealing with the problem. They modified the formulation of the economic lot-scheduling problem to adapt it for the finite, discrete time horizon settings. The solution to the problem was found with use of a heuristic. Brandimarte (2006) reformulated the problem as a plant-location model and used a Stochastic Programming approach (see (King and Wallace, 2010)) to deal with it. He implemented a node based multi stage scenario tree and used the time-sweep heuristic to solve the resulting problem. Tempelmeier and Herpers (2010) and later Tempelmeier (2011) used a target service level β_n, limiting the expected number of backordered items in each production cycle to model the stochastic MCLSP. Tempelmeier and Herpers (2010) used an ABC$_\beta$ heuristic to deal with the problem while Tempelmeier (2011) implemented a column generation heuristic. Helber et al. (2013) introduced $\sigma-$service level represent-

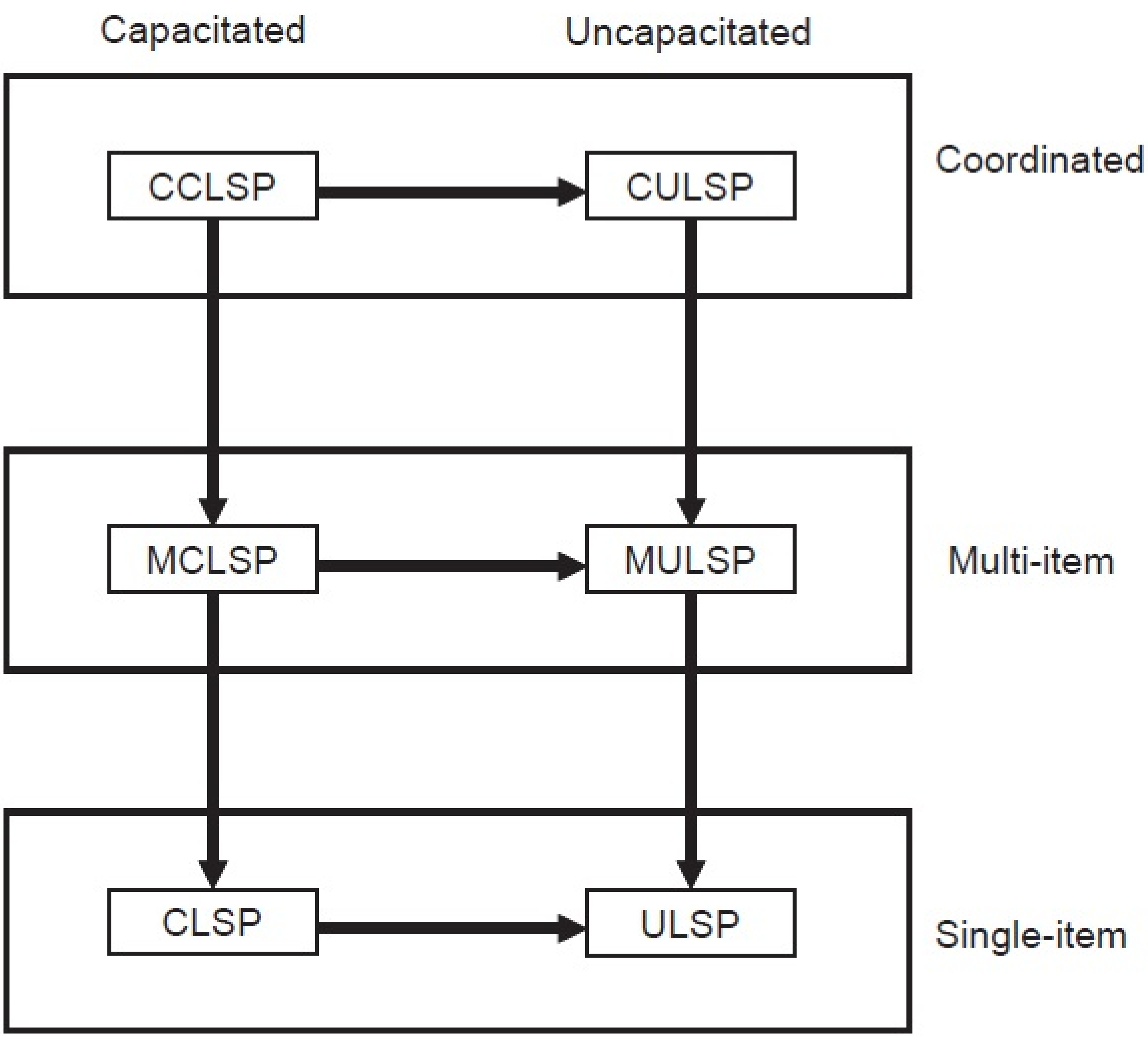

Figure 1.1: Taxonomy of lot-sizing problems by Robinson et al. (2009)

ing the expected percentage of the maximum possible demand-weighted waiting time that the customers of product n are protected against. A fix-and-optimize heuristic was used to solve the problem. The same type of heuristic was used by Tempelmeier and Hilger (2015) to solve the problem, where non-linear functions of the expected inventory and backorders were replaced by piecewise linear functions.

Papers 3 and 4 of this thesis are dealing with the coordinated capacitated lot-sizing problem under stochastic demand (CSCLSP), previously not addressed in the literature. A Stochastic Programming approach is proposed as the solution method for the problem.

1.2 Scientific contribution and summary of papers

The present thesis consists of four papers. They are united along the core problem which is joint replenishment and coordinated lot-sizing under stochastic demand and nonlinear transportation cost function requirements. Scientific contribution and summary of the papers is given below.

Paper 1 - Minimization of Transportation and Inventory Costs in a Multi-Product Probabilistic Demand Environment

Paper 1 is devoted to the developing of a model and a solution algorithm for a modified version of the stochastic joint replenishment problem. The contribution of the paper to the literature on the topic is in explicit consideration of transportation costs versus inventory costs without significant simplification of any of them in a multi-item, stochastic demand, and heterogeneous vehicle fleet setting. The model and the solution algorithm presented in the paper are tested on real world data from Stokke AS, characterized by about 100 products, a one year business cycle and two container sizes. The computational performance of the algorithm shows the potential for practical usefulness of the algorithm.

The paper is a result of joint work with my supervisors Professor David L. Woodruff and Associate Professor Johan Oppen. The model and algorithm development and implementation is done by the author of this PhD thesis with guidance of supervisors. The paper is submitted for publication to *OR Spectrum*.

Preliminary paper results were presented at Informs Annual Meettings 2012 in Phoenix, Arizona, USA, October 12-17, 2012.

Paper 2 - Multi-stage scenario generation by the combined moment matching and scenario reduction method

This paper describes the opportunity to speed up multi-stage scenario generation by combination of the moment matching method (Høyland and Wallace, 2001) and the method for scenario reduction to approximately minimize a metric (Heitsch and Römisch, 2009). The proposed method helps to improve the scenario generation process and is used to obtain scenarios for the coordinated capacitated stochastic lot-sizing problem discussed in paper 3.

The method was developed by the author with the help of supervisors Professor David L. Woodruff and Associate Professor Johan Oppen. The paper is published in the journal *Operations Research Letters*, Volume No. 42 2014, pages 374-377. The paper was also presented at the conference LOT- Logistics, optimization and transportation 2014, September 1-2, in Molde, Norway.

Paper 3 - A Stochastic Programming Approach to solve a Coordinated Capacitated Stochastic Dynamic Demand Lot-Sizing Problem with Emergency Supplies

The paper addresses a Coordinated Capacitated Stochastic Dynamic Demand Lot-Sizing Problem previously not discussed in the literature. We have developed a mathematical model to deal with the problem and solved it for a real world case from Stokke using the Stochastic Programming approach (King and Wallace, 2010). To test the advantage of the stochastic model with respect to a deterministic model, the production plan in a rolling horizon setting was applied. The total cost of the solution for the stochastic model was 51.4% lower than the one for the deterministic model.

The paper uses the scenario generation method discussed in paper 2 to implement the Stochastic Programming solution approach. Discussiones with Professor David L. Woodruff and Associate Professor Johan Oppen helped to formulate the final model and summarize the results. TThe paper is submitted to *International Journal of Logistics Systems and Management*.

An early version of the paper was presented at 2013 INFORMS Computing Society Conference at Santa Fe, New Mexico, USA, January 6-8, 2013. The model was presented at Informs Annual Meettings 2013 in Minneapolis, USA, October 6-9, 2013.

Paper 4 - Stochastic capacitated lot-sizing problems: a review of models and solution methods

The paper discusses recent advances in modeling and solution algorithms for the stochastic capacitated lot-sizing problem. Despite the particular importance of practically implementable solution methods for the problem, it has not been significantly analyzed in the literature. Aiming to fill in the gap in literature, the author developed a formulation for the general version of the Coordinated Capacitated Stochastic Lot-Sizing Problem, not covered in the literature.

1.3 Future research

Based on the contribution of the research presented in this thesis, the author suggest development of models and algorithms for the the Coordinated Capacitated Stochastic Lot-Sizing Problem as the main direction for the future research. The lack of literature on the problems with non-stationary stochastic demands can be overcome with the broader implementation of the Stochastic Programming approach.

Bibliography

Arrow, K., Harris, T., and Marschak, J. (1951). Optimal inventory policy. *Econometrica*, 19:250–272.

Atkins, D. and Iyogun, O. (1988). Periodic versus 'can-order' policies for coordinated multi-item inventory systems. *Management Science*, 34:791–796.

Balintfy, J. L. (1964). On a basic class of multi-item inventory problems. *Management Science*, 10 (2):287–297.

Brandimarte, P. (2006). Multi-item capacitated lot-sizing with demand uncertainty. *Internarional Journal of Production Research*, 44(15):2997–3022.

Harris, F. (1913). How much stock to keep on hand. *Factory: The Magazine of Management*, 10:240–241, 281–284.

Heitsch, H. and Römisch, W. (2009). Scenario tree reduction for multistage stochastic programs. *Computational Management Science*, 6, No 2:117–133.

Helber, S., Sahling, F., and Schimmelpfeng, K. (2013). Dynamic capacitated lot sizing with random demand and dynamic safety stocks. *OR Spectrum*, 35:75–105.

Høyland, K. and Wallace, S. (2001). Generating scenario trees for multistage decision problems. *Management Science*, 47, No.2:295–307.

Johansen, S. and Melchiors, P. (2003). Can-order policy for the periodic-review joint replenishment problem. *Journal of the Operational Research Society*, 54:283–290.

Khouja, M. and Goyal, S. (2008). A review of the joint replenishment problem literature: 1989-2005. *European Journal of Operational Research*, 186:1–16.

King, A. and Wallace, S. (2010). *Modelling with Stochastic Programming*. Springer.

Melchiors, P. (2002). Calculating can-order policies for the joint replenishment problem by the compensation approach. *european Journal of Operations Research*, 141:587–595.

Ozkaya, B., Gurler, U., and Berk, E. (2006). The stochastic joint replenishment problem: A new policy, analysis, and insights. *Naval Research Logistics*, 53:525–546.

Pantumsinchai, P. (1992). A comparison of three joint ordering inventory policies. *Decision Science*, 23:111–127.

Robinson, P., Narayanan, A., and Sahin, F. (2009). Coordinated deterministic dynamic demand lot-sizing problem: A review of models and algorithms. *Omega*, 37:3–15.

Sox, C. R., Jackson, P. L., Bowman, A., and Muckstadt, J. A. (1999). A review of the stochastic lot scheduling problem. *Internationa Journal of Production Economics*, 62(3):181–200.

Sox, C. R. and Muckstadt, J. A. (1999). Optimization-based planning for the stochastic lot-scheduling problem. *IIE Transactions*, 29(5):349–357.

Tempelmeier, H. (2011). A column generation heuristic for dynamic capacitated lot sizing with random demand under a fill rate constraint. *Omega*, 39:627–633.

Tempelmeier, H. and Herpers, S. (2010). ABC_β - a heuristic for dynamic capacitated lot sizing with random demand under a fill rate constraint. *International Journal of Production Research*, 48:5181–5193.

Tempelmeier, H. and Hilger, T. (2015). Linear programming models for a stochastic dynamic capacitated lot sizing problem. *Computers & Operations Research*, 59:119–125.

Viswanathan, S. (1997). Periodic review (s,s) policies for joint replenishment inventory systems. *Management Science*, 43:1447–1454.

Zheng, Y. and Federgruen, A. (1991). Finding optimal (s,s) policies is as simple as evaluating a singe policy. *Operations Research*, 39:654–665.

Paper 1

Minimization of Transportation and Inventory Costs in a Multi-Product Probabilistic Demand Environment

Minimization of Transportation and Inventory Costs in a Multi-Product Probabilistic Demand Environment

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Abstract

This paper is devoted to the optimization of logistics costs, trading-off transportation costs against inventory costs, in a setting of a multiple product flow on a single link. The problem is to determine, for each of the products, the safety stock level and a common shipping frequency during a continuous time horizon. This paper contributes to the literature on the topic by explicit consideration of transportation costs as a part of the total cost function. As an example, we use a problem faced by the Norwegian company Stokke, which designs and distributes furniture and equipment for children. Part of Stokke's supply chain is used as a source of real world data for model testing. This example is characterized by about 100 products, a one year business cycle and two container sizes. The results of the algorithm implementation on a set of simulated data shows that the algorithm performance is dependent on the data, but that runtimes are tractable for instances with up to 100 units. The implementation of the algorithm on the real world data from Stokke shows the potential for practical usefulness of the algorithm.

Keywords: Transportation; Inventory; Nonlinear Optimization; Decomposition

1 Introduction

Replenishment systems designed to minimize inventory costs are quite well developed and known in the business world. Meanwhile, the increase in transportation distances between production facilities and distribution centers has lead to an increased focus on transportation costs minimization, which often conflicts with the goal of inventory costs

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minimization. Such a problem is also closely related to the problem of coordinating distribution and production (Boudia, 2008).

This paper is devoted to the optimization of logistics costs, trading-off transportation costs against inventory costs, in a setting of a multiple product flow on a single link. In this system, a set of products with stochastic demand is shipped through a single direct link from a producer to a distribution center where the products are stored. The transportation can be performed using transport units of different capacity, but with the same constant lead time. The problem is to determine, for each of the products, the shipping frequency, order quantity, safety stock level and number and size of transport units needed to perform the delivery during a finite time horizon. As an example, we use a problem faced by the Norwegian company Stokke, which designs and distributes furniture and equipment for children. Part of Stokke's supply chain is used as a source of real world data for model testing. The scope of this example includes about 100 products, a one year business cycle and two container sizes.

Though the field of inventory management is heavily explored (Williams and Tokar, 2008), including works on cooperation between buyer and seller (Lin, 2010; Boute, 2007), and quite a few works are concerned with combined inventory management and routing problems (Andersson et al., 2010), only a few authors have studied the simultaneous minimization of transportation and inventory costs without significant simplification of either of them. Many authors mainly consider the transportation part of the problem, significantly simplifying the inventory optimization part. One of the first papers devoted to a discussion of transportation and inventory costs together is (Burns et al., 1985). The paper analyzes distribution strategies based on the total cost of transportation and inventory keeping. Speranza and Ukovich (1994) examined the influence of shipment frequencies on the transportation and inventory costs in a situation when several products are shipped via a single link. They based their study on inventory costs consisting only of handling costs. These ideas were further developed in the paper of Bertazzi and Speranza (1999). The model was extended by the introduction of a set of intermediate nodes. Inventory can be kept at all nodes including supplier and intermediate nodes. Ben-Khedher and Yano (1994) developed a multi-item joint replenishment model with consideration of transportation and proposed a heuristic to solve it. They considered holding costs as the only inventory costs. Bertazzi (2008) examined direct shipping policies with discrete shipping times. The work was based on a simplified inventory holding cost function. Berman and Wang (2006) have developed a combined model for transportation and inventory cost minimization in a multi-supplier-multi-consumer environment. Bahloul et al. (2010) examined the combined transportation and inventory cost minimization problem using an extended function of inventory costs. Their model included ordering, handling and back-order costs. To avoid problems with non-linearity in the model, the authors introduced a constant service level.

Gupta (1992) considered a model where ordering, handling and transportation costs were considered explicitly. He provides an algorithm to find optimal order quantity minimizing the total logistical costs of the system, consisting of one product and a homogeneous transportation fleet. His ideas were further developed by Madadi et al. (2010). They based their research on an (r,S) inventory replenishment system and included fixed and variable transportation costs. The system they were dealing with also included only one product and a homogeneous transport fleet. An approximation method was used to predetermine the safety factor. Zhao et al. (2004) considered an inventory problem including production, ordering, holding and transportation costs, where transportation costs included fixed and variable parts. This model, as well as others, included only one product and a homogeneous transportation fleet.

Taking into consideration multiple items adds significant difficulties. One of the first models designed to manage coordinated replenishment from a single supplier to a single consolidation point was presented by Balintfy (1964). This model is widely known as a “can-order” system. This model did not provide an optimal inventory policy. Simmons (1972) presented an optimal inventory policy under a hierarchy of setup costs and developed an algorithm capable of giving a near optimal solution for this policy. His algorithm was improved by Hartfiel and Curry (1974) to give an optimal solution. According to Aksoy and Erenguk (1988) and Silver et al. (1998) most authors worked with periodic review multi-item replenishment systems.

Since the 1960s, the problem of joint replenishment has been heavily explored. Khouja and Goyal (2008) presented a review of the joint replenishment problem literature until 2005. According to them, most authors by 2005 concentrated on finding optimal and sub-optimal solutions for classical JRP. Khouja and Goyal indicate some papers issued after 2005, where authors are dealing with constrained versions of JRP, including storage, transportation, budget and other restrictions. Hoque (2006) considered the joint replenishment problem with storage and transport capacities in deterministic demand settings.

Qu et al. (1999) combined the ideas of shipment consolidation theory and JRP models in order to develop a model capable of optimizing the costs of an integrated inventory-transportation system for multiple items. One of their assumptions is unlimited transportation capacity. A heuristic approach was used to solve the problem, and a lower bound to the optimal solution was given. Wang et al. (2013) modified the problem discussed by Qu et al. into a multi objective stochastic JRP. Instead of considering the costs of back-orders as a part of the total cost function, they modeled another objective of minimizing the number of backlogged items. A heuristic was used to solve the problem, assuming unlimited transportation capacity.

Another area of study, closely related to the problem discussed in this paper, is shipment consolidation. Çetinkaya (2005) discussed several integrated policies for stochastic problems, all of which assume negligible lead times and uncapacitated vehicles. Kiesmüller and de Kok (2005) combined the ideas of shipment consolidation, assuming target service level and no vehicle capacity restrictions in a multi-item multi-echelon inventory system. Kiesmüller (2010) presented the JRP problem with shipment consolidation under assumptions of full truck load and targeted service level in stochastic settings.

The model discussed in this paper is based on the periodic review replenishment system, determining the optimal pair of review period length r and upper level of inventory S. The detailed description of this type of model can be found in Silver et al. (1998). The aim of the model is to represent a replenishment system capable to fulfill the task of demand satisfaction with minimal logistical costs, including ordering, transportation, inventory handling and back-order costs.

Our research differs from previous works in one or more of the following ways: we consider multiple items with normally i.i.d demands in a stochastic setting, a single-echelon inventory system is analyzed, a heterogeneous fleet where an unlimited number of vehicles of different sizes are used for transportation, both full truck and less than full truck loads are allowed, neither the replenishment cycle length nor the service level for any of the items are predetermined and fixed, and the solution algorithm finds an optimal solution for a given problem.

The problem of trading-off holding costs against ordering (transportation) costs was first raised by F.Harris (1913). Considering multiple items authors typically include fixed cost to place an order and an item-dependent part of ordering costs (Balintfy, 1964; Hartfiel and Curry, 1974). According to Khouja and Goyal (2008), if a certain item is included in the order then a fixed amount is added to the major ordering costs. Thus ordering costs are independent on the quantity of a given item included in the order. In this paper we distinguish transportation costs from ordering costs. Ordering costs include all kind of clerical costs and are fixed, while transportation costs include inbound logistics costs and depend on the ordered transportation capacity, which should fit to an order. In this setting transportation costs have a piecewise linear structure. We optimize total logistical costs by trading-off the size of ordered capacity against the amount of inventory held on stock. One can save on transportation costs, ordering larger transportation capacity. On the other hand, savings on transportation are extreme only if capacity is utilized completely. Hence the order should be large, leading to the higher holding costs. When transportation costs are balanced with holding costs the overall logistical costs are minimized.

The contribution of the current paper to the literature on the topic is in explicit consideration of transportation costs versus inventory costs without significant simplification of any of them in a multi-item, stochastic demand, and heterogeneous vehicle fleet setting. Most authors consider transportation costs as a linear function of product quantity by adding minor per unit ordering costs to the objective function. We argue that many wholesalers use third party logistics companies to perform transportation, thus paying a fixed price for a unit of given capacity on a route, and not for a unit of product. Thus transportation costs in our paper have a piecewise linear structure and are dependent on the number and quantity of product indirectly. Some other features of the proposed model, such as a common review period, certain level of transportation capacity buffer, the type of backorder costs among others, are driven by the restrictions from the real world case from Stokke. The proposed model is optimally solved both for simulated and real case data, showing significant improvement compared to the other model traditionally used for a problem.

The paper is organized in the following way: Section 2 is devoted to the description of the integrated mathematical model. Section 3 presents development of several algorithms capable of finding optimal solutions based on the given model. Section 4 presents the results of the algorithm implementation on a given problem together with the discussion of possible extensions of the model or modifications of the algorithm. Section 5 summarizes the findings and presents conclusions.

2 An Integrated Logistics Model

A buyer maintains inventory of $|\mathcal{N}|$ different products, which are all ordered from a single supplier. The demands for all products are assumed to be i.i.d. and follow normal distribution $\sim N(D_n; \sigma_n)$, where D_n and σ_n are expected yearly demand and standard deviation of yearly demand for product $n \in \mathcal{N}$ respectively. Any of $|\mathcal{M}|$ transport unit sizes can be used to perform the transportation. There is no limit on the number of available units of each capacity. All products are assumed to be ordered simultaneously based on an *(r,S)* periodic review replenishment system. This means that reviews are performed $\frac{1}{r}$ times during time horizon and at each review all the products are ordered up to the specified level S_n. Many authors (Silver et al., 1998) assume multiple review periods when using a periodic review replenishment policy in multiple items environment. A common review period is assumed in our model. There could be a few reasons to do that in practice:

- Items in a group have approximately the same relative standard deviation of demand $\frac{\sigma_i}{D_i} = \frac{\sigma_j}{D_j}$. Then it is optimal to review these items simultaneously
- The cost of a review does not depend on the number of items reviewed
- Managerial aspects, such as contract restrictions or error elimination in order processing, etc.

In case of Stokke, the main reason to have the common review period for all items is a managerial restriction. In order to decrease the mismatch between planned review and actual review for each particular item, company decided to have a common review period for all of them. Beside, since the cost of review does not depend on the number of items in the review, this will not add extra costs, even if it will be found out, that there is no need to order a particular item in a particular period.

The order-up-to level for any of the items is defined based on the well-known formula from Silver et al. (1998):

$$S_n = D_n^{L+r} + k_n \cdot \sigma_n^{L+r} \tag{1}$$

where D_n^{L+r} is the expected demand for product n during the review period r and the lead time L, k_n is the safety factor for product n, and σ_n^{L+r} is the standard deviation of demand for product n during the risk period.

The replenishment system presented in this paper accounts for four types of logistics costs: ordering costs - costs of checking the inventory, issuing and following the order till its fulfillment; holding costs - costs of keeping the inventory in a warehouse; back-order costs - penalties for inability to fulfill the customer's order on time; and transportation costs - costs associated with the necessary transportation capacity. To manage this system one is interested in finding a pair *(r,S)* for each of the products in such a way that the total costs are minimized and the review period length is the same for all products.

Since the problem discussed here is inspired by a real world case, it has a set of specific features. Nevertheless, such features of the problem are common for many companies. Thus the presented model is suitable for practical use. The most striking difference between the model and those common in the academic literature is the type of back-order cost used. We assume a fixed charge for occurrence of back-order of each of the items. We analyze the inventory problem of a wholesaler, so whenever he receives an order which cannot be satisfied in full, the wholesaler ships the order without the missing items. When the missing items are back in stock, they are immediately shipped to the customer. According to the contract between the wholesaler and its customers, there are no penalties charged for delay of shipment of absent items disregarding the quantity, and the length of shortage. Nevertheless, an extra shipment leads to an increase in outbound transportation costs. The average of this costs is used as the cost per shortage. Since both inbound and outbound transportation are ordered from a third party logistics company (companies), the need to make an extra transportation to the customer will not restrict the amount of vehicles available for inbound transportation.

We also assume that the ordering costs depend only on the number of orders per time unit, and does not depend on inclusion of a certain product in the order. Whenever an order is placed, there is a need to review each item, disregarding whether the item will be included in the order or not, thus the cost of placing an order stays constant. It is worth mentioning that at Stokke, the warehouse is operated by a third party logistics company, hence the handling (unloading and inspection) costs are fixed according to the contract, which corresponds to fixed ordering costs.

Another assumption is that the transportation costs are charged per transportation unit of a given size, and does not depend on the transportation unit load. This assumption is often true in cases where a company orders a long haul transport from a third party. In addition, the price per transportation unit typically does not change depending on the total number of units ordered.

Table 1: Notation

Sets	
$\mathcal{N}$	= set of products
$\mathcal{M}$	= set of transport units size indexes
Parameters	
D_n	= expected value of yearly demand for product n , $n \in \mathcal{N}$
σ_n	= standard deviation of annual demand for product n , $n \in \mathcal{N}$
V_n	= volume of the product unit n, $n \in \mathcal{N}$
U_m	= capacity of the transport unit of size index m, $m \in \mathcal{M}$
E_m	= transportation costs per transport unit of size index m, $m \in \mathcal{M}$
W_n	= unit costs for product n, $n \in \mathcal{N}$
A	= ordering costs per order
H	= annual holding costs as a fraction of unit costs
B_n	= back-order costs per occurrence of product n shortage, $n \in \mathcal{N}$
L	= length of the lead time as a fraction of a year
α	= probability that the transportation capacity will not be exceeded
Variables	
r	= the length of the review period as a fraction of a year
k_n	= safety factor for product n, $n \in \mathcal{N}$
y_m	= number of transport units of size index m used in one replenishment, $m \in \mathcal{M}$

Table 1: (continued)

Stochastic Parameter	
$o_n(r)$	= amount of product n ordered in a review period , $n \in \mathcal{N}$
Functions	
$\Phi(x) = 1 - P_{u\geq}(x)$	= cumulative probability function of standard normal distribution

Given these assumptions together with those mentioned before, we propose the model presented in Table 2 that makes use of parameters and variables shown in Table 1. Solving this model to optimality will allow us to find minimal total costs, and, corresponding to them, optimal (r, S_n) pairs. While solving the model one will also find the expected number of vehicles of each size, y_m, to be ordered in each review period. However, since the demand is varying, the number of transport units used can differ from period to period, and thus has to be decided on a later (operational) stage.

Table 2: Combined model for minimization of annual transportation and inventory costs

$$min\, C = \frac{1}{r} \cdot A + \sum_{n \in \mathcal{N}} \left(\frac{D_n \cdot (r+L)}{2} + k_n \cdot \sigma_n \cdot \sqrt{(r+L)} \right) \cdot W_n \cdot H + \\ + \frac{1}{r} \cdot \sum_{m \in \mathcal{M}} y_m \cdot E_m + \frac{1}{r} \cdot \sum_{n \in \mathcal{N}} P_{u\geq}(k_n) \cdot B_n \quad (2)$$

Subject to:

$$Prob \left(\sum_{n \in \mathcal{N}} o_n(r) \cdot V_n \leq \sum_{m \in \mathcal{M}} y_m \cdot U_m \right) \geq \alpha \quad (3)$$

$$r > 0 \quad (4)$$

$$k_n \geq 0, \forall\, n \in \mathcal{N} \quad (5)$$

$$y_m \in \mathbb{N}, \forall\, m \in \mathcal{M} \quad (6)$$

The objective function (2) is the total expected costs which consist of ordering costs, inventory holding costs, shortage costs and transportation costs. Ordering costs are dependent on the number of replenishments per year. As all the products are ordered simultaneously, the cost of issuing one order is independent of the products included in the order and is constant. The number of replenishments is the same for all products.

Holding costs are dependent on the average inventory level and the chosen level of safety stock. The average inventory level in turn depends on the length of the review period. Fewer review points during the year (i.e., a long review period) results in higher average levels of inventory for each of the products. The safety stock level is determined by a combination of the safety factor value and the review period length. The third term in the objective function is transportation costs, which are dependent on the number of transport units of each size needed to transport the order. Finally, back-order costs represent the expected penalty which is paid for each occurrence of unsatisfied demand. This depends on the total number of orders per year and the cumulative probability of the standard normal distributed variable u being larger then the safety factor k.

The objective function is subject to only one operational constraint (3). It is presented as a chance constraint, which requires that the ordered transportation capacity is capable to carry an order with at least α probability. In this case $o_n(r)$ denotes stochastic order during review period r. Since the demand for all the products is i.i.d. normal, there are no lost sales, and an order-up-to policy is used, the order size for each of the products will be i.i.d. normal, i.e., $\sim N(r \cdot D_n,\ r \cdot \sigma_n)$.

Constraint (3) together with the minimization objective function leads to a piece-wise solution function. Other constraints (4)-(6) impose non-negativity requirements on decision variables, and integrality requirements on the number of transport units of each size. It can be seen that constraint (3) is not linear, thus we will rearrange it to form a linear constraint. Because of the random order sizes, o_n, the constraint expression can be viewed as a random variable. We will rewrite the constraint expression in the following way:

$$\sum_{n \in \mathcal{N}} o_n \cdot V_n - \sum_{m \in \mathcal{M}} y_m \cdot U_m \leq 0 \tag{7}$$

Assuming that the order for each individual item is an independently distributed random variable following normal distribution, the left hand side of the constraint expression is also a random variable following normal distribution:

$$\sim N\left(r \cdot \sum_{n \in \mathcal{N}} D_n \cdot V_n - \sum_{m \in \mathcal{M}} y_m \cdot U_m\,,\ r \cdot \sqrt{\sum_{n \in \mathcal{N}} \sigma_n^2 \cdot V_n^2} \right) \tag{8}$$

Hence, chance constraint (3) can be rewritten in the following way:

$$\frac{0 - \left(r \cdot \sum_{n \in \mathcal{N}} D_n \cdot V_n - \sum_{m \in \mathcal{M}} y_m \cdot U_m \right)}{r \cdot \sqrt{\sum_{n \in \mathcal{N}} \sigma_n^2 \cdot V_n^2}} \geq \Phi^{-1}(\alpha)\,, \tag{9}$$

where $\Phi^{-1}(\alpha)$ is the quantile function of a standard normal distribution for α probability level. By re-arranging (9) we get:

$$r \cdot \sum_{n \in \mathcal{N}} D_n \cdot V_n + \Phi^{-1}(\alpha) \cdot r \cdot \sqrt{\sum_{n \in \mathcal{N}} \sigma_n^2 \cdot V_n^2} \leq \sum_{m \in \mathcal{M}} y_m \cdot U_m \tag{10}$$

Here we need to notice that we assume that for each given order the number of transport units is ordered separately. Thus, in case when the predetermined transport capacity will not fit all the units of a given order, some extra transport units will be ordered. Since the demand distribution is stationary, in the long run there will appear situations when the predetermined capacity is excessive, thus some of the transport units will not be used and paid for. Hence the expected cost of the system will not change. Moreover, the assumption of stationary demand means that, if we analyze an infinite horizon, on average we will need to transport D_n units of each product n during the time horizon. So, in the long run, the model overestimates the expected transportation costs, since we assume that the transport capacity will manage to transport both the average volume of demand $\sum_{n \in \mathcal{N}} D_n \cdot V_n$ and some extra volume $\Phi^{-1}(\alpha) \cdot \sqrt{\sum_{n \in \mathcal{N}} \sigma_n^2 \cdot V_n^2}$ of products. The level of capacity buffer is determined by α, and is set by the contract with the third party logistic provider. Hence one of the purposes of the model being solved is to determine the range of possible capacity variation within one replenishment period. However, it does not mean that the maximum level of transportation capacity is to be paid for. In contrast, the case company can vary the capacity without extra costs.

The solution function (constrained objective function) is non-convex and discontinuous. To illustrate the shape of the solution function we will use a simplified version of the problem:

1. There is only one product analyzed in the replenishment system.

2. Units of only one size are used to transport the orders.

The model can be then written in the following way:

$$min\, C = \frac{1}{r} \cdot A + \left(\frac{D \cdot (r+L)}{2} + k \cdot \sigma \cdot \sqrt{(r+L)} \right) \cdot W \cdot H + \\ + \frac{1}{r} y \cdot E + \frac{1}{r} \cdot P_{u \geq}(k) \cdot B \quad (11)$$

subject to:

$$r \cdot D \cdot V + \Phi^{-1}(\alpha) \cdot r \cdot \sigma \cdot V \leq y \cdot U \quad (12)$$

The solution function for this model presents a set of banana-shaped surfaces (Figure 1) disconnected in points, where

$$r \cdot D \cdot V + \Phi^{-1}(\alpha) \cdot r \cdot \sigma \cdot V = X \cdot U,\ X = 1, 2.., \left\lceil \frac{D \cdot V + \Phi^{-1}(\alpha) \cdot \sigma \cdot V}{U} \right\rceil, \quad (13)$$

where X is a parameter, "chosen" from the domain of y. X denotes the number of transportation units used in one review period.

In other words, for each given number of transport units there is a separable surface with its unique local optimum. The global optimum should be found among these local optima.

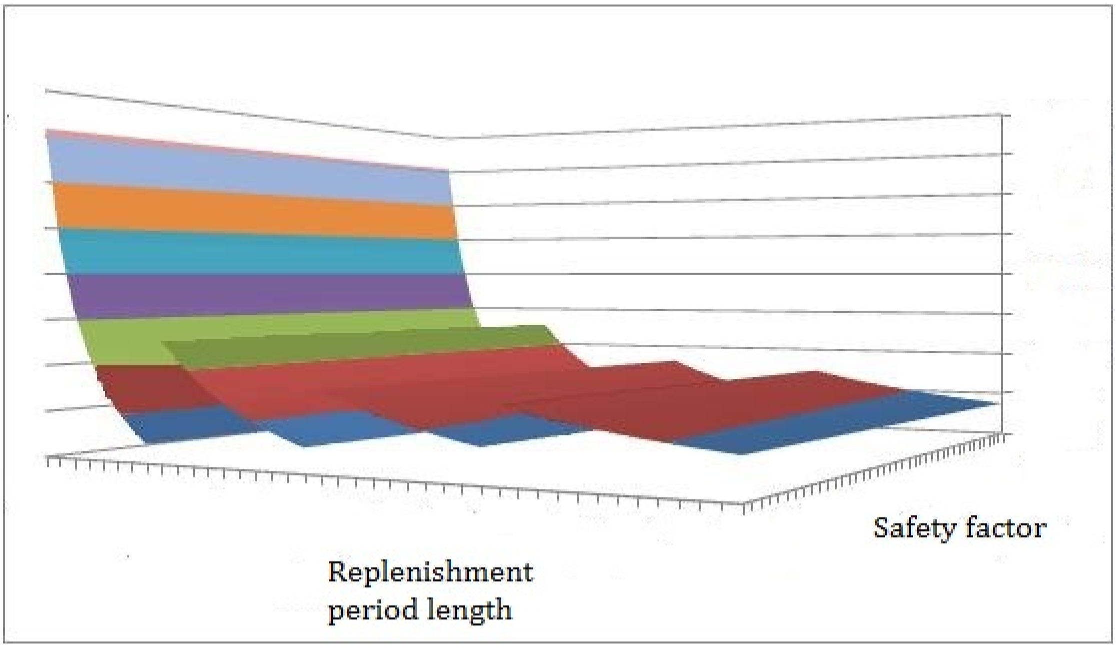

Figure 1: Simplified Solution Function

If the number of transport units is fixed then the solution function of the model, presented by equations (11-12), is convex and continuous on the (*r,k*) space, according to Whitin (1953). Moreover this function can be easily extended to handle multiple items, and the extended function is also convex on the (r, k_n) space, since the sum of two convex functions is also convex. Thus one can find a number of nonlinear solvers capable to deal with the problem. Hence our task is to construct bounds for the number of transport units of each size.

3 Solution algorithms for problem variants

We now present a set of algorithms to provide optimal solutions for different types of the problems presented above. They will include algorithms for the following models: Multiple Items One Transport Unit Size model; Multiple Items Multiple Transport Unit Sizes model; and a specific case of Multiple Items Two Container Sizes model.

The first algorithm is developed for the Multiple Items One Transport Unit Size problem:

1. Define the domain of y: $y = \left[1, \ldots, \left\lceil \frac{\sum_{n\in\mathcal{N}} D_n \cdot V_n + \Phi^{-1}(\alpha)\sqrt{\sum_{n\in\mathcal{N}} \sigma_n^2 \cdot V_n^2}}{U} \right\rceil\right]$, where the maximum number of transport units is defined based on the expected total volume of all goods ordered during the time horizon. So, if the time horizon is one year, the maximum number of transport units shall fit such number of products, which will satisfy the total yearly demand.

2. Decompose the solution function based on each given value of *y*.

3. Find the minimum of the objective function for each given value of y using any available non-linear solver. The Ipopt (Wächter and Biegler, 2006) solver was used in

this paper to get the optimal solution for each segment of the decomposed solution function using a model written in Pyomo (Hart et al., 2012). Let's call these solutions local minima.

4. Find the global minimum of the objective function among all local minima.

Algorithm 1 *Algorithm's pseudo code for Multiple Items One Transport Unit Size Problem*

1: Calculate $y^{max} = \left\lceil \frac{\sum_{n \in \mathcal{N}} D_n \cdot V_n + \Phi^{-1}(\alpha)\sqrt{\sum_{n \in \mathcal{N}} \sigma_n^2 \cdot V_n^2}}{U} \right\rceil$.
2: Solve model (2)-(6) fixing variable y to 1; Set $C^* = C(1)$, record solution
3: **for** $y = \{2, \ldots, y^{max}\}$ **do**
4: Solve model (2)-(6) fixing variable y to current value
5: **if** $C^* \geq \hat{C}(y)$ **then**
6: Set $C^* = \hat{C}(y)$; record solution
7: **end if**
8: **end for**

Algorithm 1 shows the simple method for computing the solution with multiple items, but only one transport unit size.

As the value of variable y from the model presented in (2)-(6) is fixed in each iteration (lines 2 and 4), the search space becomes convex so sub-problems can be solved to optimality with the Ipopt non-linear solver. The next algorithm is developed for the most general Multiple Items Multiple Transport Unit Sizes problem:

1. Order transport unit sizes in increasing order of unit load costs. I.e., if $\frac{E_i}{U_i} \leq \frac{E_j}{U_j}$, then y_i should be considered before y_j.

2. Define the domains of y_m:

   (a) For the transport unit size with the lowest unit load cost,

   $y_1 = \left[0, \ldots, \left\lceil \frac{\sum_{n \in \mathcal{N}} D_n \cdot V_n + \Phi^{-1}(\alpha)\sqrt{\sum_{n \in \mathcal{N}} \sigma_n^2 \cdot V_n^2}}{U_1} \right\rceil \right]$. The maximum number of transport units of size index 1 is defined based on the total volume of all goods. So the maximum number of transport units of the cheapest size shall fit yearly demand for all products simultaneously.

   (b) For all others find the least common multiplier of y_m and y_{m-1} capacities $lcm(U_m, U_{m-1})$. Then $y_m = \left[0, \ldots, \frac{lcm(U_m, U_{m-1})}{U_m} - 1\right]$. In other words, one should first find the least common multiplier for capacities of transport units with current and previous size index. Then divide it by the capacity of transport unit with current size index and subtract 1. For example, if we have 10 and 25 foot trucks and the 25 foot truck has the cheapest unit load, then we at most will use four 10 foot trucks. The reason is, that five 10 foot trucks can be replaced with two 25 foot trucks which will cost less.

3. Decompose the solution function based on each given value of y_m. So now the total number of decomposed solutions will be equal to $\left\lceil \frac{\sum_{n\in\mathcal{N}} D_n\cdot V_n+\Phi^{-1}(\alpha)\sqrt{\sum_{n\in\mathcal{N}}\sigma_n^2\cdot V_n^2}}{U_1}\right\rceil \cdot \prod_{m\in\mathcal{M}\setminus\{1\}}\left(\frac{lcm(U_m,U_{m-1})}{U_m}-1\right)$.

4. Find minimum of the objective function for each given value of y_m.

5. Find the global minimum of the objective function among all local minima.

Algorithm 2 *Algorithm's pseudo code for Multiple Items Multiple Transport Unit Sizes Problem*

1: Order $y_m \quad \forall m \in \mathcal{M}$ in ascending order of $\frac{E_m}{U_m}$.
2: Calculate $y_1^{max} = \left\lceil \frac{\sum_{n\in\mathcal{N}} D_n\cdot V_n+\Phi^{-1}(\alpha)\sqrt{\sum_{n\in\mathcal{N}}\sigma_n^2\cdot V_n^2}}{U_1}\right\rceil$.
3: **for** $y_m, \forall m \in \mathcal{M}\setminus\{1\}$ **do**
4:     Calculate $y_m^{max} = lcm\left(U_m, U_{m-1}\right) - 1$
5: **end for**
6: Let $\mathcal{Y}$ be the set that enumerates all $\hat{y}$ vectors with elements y_m from 0 to y_m^{max}
7: Let $\hat{y}^0$ be the vector where all elements $y_m = 0$
8: Set $C^* = \infty$
9: **for** each $\hat{y} \in \mathcal{Y}\setminus\{\hat{y}^0\}$ **do**
10:     **if** $C^* \geq \hat{C}\left(\hat{y}\right)$ **then**
11:         Set $C^* = \hat{C}\left(\hat{y}\right)$; record solution
12:     **end if**
13: **end for**

Algorithm 2 establishes upper bounds on the number of transport units of all sizes.

As an example, let us again consider the situation with 10 and 25 foot trucks, when a 25 foot truck has the cheapest unit load. Assume that five 25 foot trucks can carry the total volume of yearly demand, so the same volume can be transported by thirteen 10 foot trucks. This means that during some replenishment period we can use from zero to thirteen 10 foot trucks and from zero to five 25 foot trucks. While all possible combinations of transport capacities are equal to $14 \cdot 6 = 84$, the algorithm we propose limits this amount to $5 \cdot 6 - 1 = 29$, cutting off combinations which will not improve the objective function.

The upper bounds for all variables y_m are set up in lines 1-5 of algorithm 2. In line 6 a set is created that enumerates all possible combinations of available transport units of different sizes. The number of transport units of each size ordered in one replenishment, y_m, can be from 0 to y_m^{max}. So $\mathcal{Y}$ will have $\prod_{m\in\mathcal{M}} y_m^{max}$ elements. Then in line 7 the algorithm establishes a vector $\hat{y}^0$, where each element $y_m = 0$, i.e., $\hat{y}^0 = (0,\, 0,\, ...,\, 0)$.

In line 8 the algorithm establishes an initial reference solution value, which is equal to a sufficiently large number. In practice, the initial solution can be calculated as a function $\hat{C}\left(\hat{y}\right)$ of any vector $\hat{y}$ realization, except vector $\hat{y}^0$. The function $\hat{C}\left(\hat{y}\right)$ is defined here as an optimal solution of the model defined in equations (2)-(6) with the vector of variables y fixed to vector $\hat{y}$.

Lines 9-12 presents code to solve model (2-6) for each realization of vector $\hat{y}$ except vector of all zeros and to set the reference solution to the best of found solutions $\hat{C}(\hat{y})$. The final reference solution is globally optimal for the problem.

Algorithm 3 *Algorithm's Pseudo code for Multiple Items Two Container Sizes Problem*

```
1: Calculate y_{40'}^{max} = ⌈ (Σ_{n∈N} D_n·V_n + Φ^{-1}(α) √(Σ_{n∈N} σ_n^2·V_n^2)) / U_{40'} ⌉.
2: Solve model (2)-(6) fixing y_m to 1, ∀m ∈ M; Set C* = C(1); record solution
3: for y_{40'} = {0, ..., y_{40'}^{max}} do
4:     for y_{20'} = {0, ..., 1} do
5:         Solve model 2 fixing variable y_m, ∀m ∈ M to current value
6:         if C* ≥ Ĉ(ŷ) then
7:             Set C* = Ĉ(ŷ); record solution
8:         end if
9:     end for
10: end for
```

Note that procedures to set the upper bound for the number of transport units with size index greater than 2 as given in step 4 can be improved upon in some situations. To decrease the number of decomposed solutions, i.e. the number of realizations of vector Y, one can, for example, look for the least common multipliers of the capacities of current size index and all previous size indexes. Then the upper bound on the number of units with current size index will be $y_m^{max} = min_{t \in (1,m-1)} \left(lcm\left(U_m, U_{m-t}\right) - 1 \right), \forall m \in \mathcal{M} \backslash \{1\}$. Due to a decreased number of elements in set $\mathcal{Y}$, a smaller number of optimization sub-problems will be solved, leading to an overall speed-up of the algorithm; however, we did not use this enhancement in our computational experiments.

Note also that we did not assume lower per unit transportation costs for larger sizes of vehicles. In cases when vehicles of different capacity have the same per unit cost, it does not matter which of them will have the lower order, since $lcm\left(U_m, U_{m-1}\right)$ will be the same disregarding which of the vehicles has order m and which $m-1$.

Now we will present a specific case of the algorithm for the problem, when 20 and 40 foot containers are used for transportation (Algorithm 3). This algorithm is specifically designed to deal with the problem of the case company Stokke.

4 Algorithm implementation and performance

The general algorithm (Algorithm 2), presented in the previous section was used to solve a set of simulated problems as well as a real-world problem of Stokke. The algorithm was coded using Python and Pyomo (Hart et al., 2011) and in each iteration a constrained non-linear problem was solved using the Ipopt non-linear solver (Wächter and Biegler, 2006). The CPU running times were computed for an 16x Intel Xeon(R) CPU E5620 2400 GHz processor with 11,7 GB RAM. At first the algorithms were tested on a set of simulated instances, where instance of type "2-10", for example, includes 2 transport unit sizes and 10 products to consider. 10 instances of each type were used to compute average CPU running time as well as its standard deviation. In each instance of a given type capacities of transport units, α value, transportations costs per transport unit and ordering costs stayed the same. Other parameters, such as products' demand, standard deviation of demand, unit costs and backorder costs varied from instance to instance. Table 3 shows the average computing time in seconds used by the algorithm to solve problem instances of a given type to optimality.

Table 3: CPU times for cases with products yearly demand up to 1000 units

Problem Instance Type	Average time, sec.	Standard deviation of time
2-10	21.3	4.73
2-100	5076.4	535.83
2-1000	1793614.6	52924.11
3-10	36.0	6.95
3-100	11342.4	1629.40
4-100	50068.2	5422.45

The simulation outcome showed that the algorithm performance is highly dependent on the ratio of total demand volume (weight, etc.) to the transport units capacities. In general, the more transport units with the cheapest unit load needed to transport the total yearly demand, the more time required by the algorithm. This might require a parallel implementation for companies working with a large volumes of many products. As well, if the number of vehicle sizes becomes very large, an improved procedure to find reasonable combinations of the fleet should be implemented in order to reduce the number of non-linear problems to solve.

The algorithm was also applied to solve a real world problem of Stokke using algorithm 3. The instance included 98 products and two sizes of containers (40 foot and 20 foot) to transport them. This problem instance was solved in 3317.16 seconds of CPU time. As the problem is typically solved on the tactical stage of planning and budgeting, i.e. at most once per year, such a runtime is reasonable enough to say that it is useful for business practitioners. We have also tested the performance of the proposed model against the one which is typically used in the company. The common practice is to compute the length of the review period and the values of safety factors without consideration of transportation costs. If to compute expected total logistical costs using the approach commonly used on practice, they will be 8.04 % higher compared to the expected total logistical costs found with use of proposed model. Hence, we can conclude that the model demonstrates practical usefulness for Stokke.

5 Conclusion

This paper is devoted to a problem of optimization of expected total logistical costs in a system where joint replenishment can be implemented. A mathematical model to deal with such a problem is presented together with an algorithm to find a globally optimal solution. The model proposed to deal with the problem is solved on a tactical level in order to find optimal (r, S_n) pairs, allowing minimization of expected total costs. The expected number of transport units of each size to be ordered during each review period is found as a bi-product. However, the actual number of ordered transport units can differ from period to period. Such variation is allowed by one of the assumptions indicating that the transportation is ordered through a third party logistics company. Other assumptions include heterogeneous fleet of transport units, stochastic and stationary i.i.d items' demands following normal distribution, and back-order costs per occurrence of stockout.

The results of the algorithm implementation on a set of simulated data have shown that the algorithm performance is dependent on the data but that runtimes are tractable for instances with up to 100 products. However, difference in parameters can lead to changes in running time. The main reason for that lies in the iterative structure of the algorithm. A constrained inventory optimization problem is solved for each allowed option of transport combinations. All else being equal, higher total demand volume leads to a larger number of iterations, and hence to a longer algorithm running time. Applying the algorithm on a real life case gave an 8% reduction in the expected total cost function, compared to the existing practice.

Future research can be devoted to set tighter bounds on the number of iterations, i.e., on the number of vectors $\hat{y}$. Another option to speed up the algorithm is to parallelize it. Nevertheless, the implementation of the algorithm on the real world data from the Norwegian company Stokke shows its practical usefulness.

References

Aksoy, Y. and Erenguk, S. S. (1988). Multi-item inventory models wtih co-ordinated replenishments: A survey. *International Journal of Operations and Production Management*, 8(1):63–73.

Andersson, H., Hoff, A., Christiansen, M., Hasle, G., and Løkketangen, A. (2010). Industrial aspects and literature survey: Combined inventory management and routing. *Computers & Operations Research*, 37:1515–1536.

Bahloul, K., Baboli, A., and Campagne, J.-P. (2010). A new replenishment policy based on mathematical modeling of inventory and transportation costs with probabilistic demand. In *8th International Conference of Modeling and Simulation-MOSIM'10-Hammamet-Tunisia "Evaluation and optimization of innovative production systems of goods and services"*.

Balintfy, J. L. (1964). On a basic class of multi-item inventory problems. *Management Science*, 10(2):287–297.

Ben-Khedher, N. and Yano, C. A. (1994). The multi-item joint replenishment problem with transportation and container effects. *Transportation Science*, 28(1):37–54.

Berman, O. and Wang, Q. (2006). Inbound logistics planning: Minimizing transportation and inventory cost. *Transportation Science*, 40(3):287–299.

Bertazzi, L. (2008). Analysis of direct shipping policies in an inventory-routing problem with discrete shipping times. *Management Science*, 54(4):748–762.

Bertazzi, L. and Speranza, M. (1999). Inventory control on sequences of links with given transportation frequencies. *International Journal of Production Economics*, 59:261–

270.

Boudia, M. (2008). Coordination of production planning and distribution. *4OR*, 6(1):93–96.

Boute, R. (2007). Impact of replenishment rules with endogenous lead times on supply chain performance. *4OR*, 5(3):261–264.

Burns, L., Hall, R. W., Blumenfeld, D., and Daganzo, C. (1985). Distribution strategies thath minimize transportation and inventory costs. *Operations Research*, 33:469–490.

Çetinkaya, S. (2005). *Applications of Supply Chain Management and E-Commerce Research*, chapter Coordination of Inventory and Shipment Consolidation Decisions: A Review of Premises, Models, and Justification, pages 3–51. Springer.

Gupta, O. (1992). A lot-size model with discrete transportation costs. *Computers and Industrial Engineering*, 22:397–402.

Harris, F. (1913). How much stock to keep on hand. *Factory: The Magazine of Management*, (10):240–241, 281–284.

Hart, W., Watson, J.-P., and Woodruff, D. (2011). Pyomo: Modeling and solving mathematical programs in Python. *Mathematical Programming Compuation*, 3(3).

Hart, W. E., Laird, C., Watson, J.-P., and Woodruff, D. (2012). *Pyomo - Optimization Modeling in Python*. Springer.

Hartfiel, D. and Curry, G. (1974). An algorithm for optimal inventory policies for systems with joint setup costs. *Management Science*, 20(8):1175–1177.

Hoque, M. (2006). An optimal solution technique for the joint replenishment problem with storage and transport capacities and budget constraints. *European Journal of Operations Research*, 175:1033–1042.

Khouja, M. and Goyal, S. (2008). A review of the joint replenishment problem literature: 1989-2005. *European Journal of Operations Research*, 186:1–16.

Kiesmüller, G. (2010). Multi-item inventory control with full truckloads: A comparison of aggregate and individual order triggering. *European Journal of Operations Research*, 200(1):54–62.

Kiesmüller, G. and de Kok, A. (2005). A multi-item multi-echelon inventory system with quantity-based order consolidation. Technical report, Faculty of Technology Management, Technische Universiteit Eindhoven.

Lin, Y.-J. (2010). A stochastic periodic review integrated inventory model involving defective items, backorder price discount, and variable lead time. *4OR*, 8(3):281–297.

Madadi, A., Kurz, M. E., and Ashayeri, J. (2010). Multi-level inventory management decisions with transportation cost consideration. *Transportation Research Part E*, 46:719–

734.

Qu, W., Bookbinder, J., and Iyogun, P. (1999). An integrated inventory-transportation system with modified periodic policy for multiple products. *European Journal of Operations Research*, 115(2):254–269.

Silver, A. D., Pyke, D. F., and Peterson, R. (1998). *Inventory Management and Production Planning and Scheduling. 3rd Edition*. John Wiley & Sons Ltd.

Simmons, D. (1972). Optimal inventory policies under a hierarchy of setup costs. *Management Science*, 18(10):591–600.

Speranza, M. and Ukovich, W. (1994). Minimizing transportation and inventory costs for several products on a single link. *Operations Research*, 42(5):879–894.

Wächter, A. and Biegler, L. (2006). On the implementation of a primal-dual interior point filter line search algorithm for large-scale nonlinear programming. *Mathematical Programming*, 106:25–57.

Wang, L., Qu, H., Liu, S., and Dun, C.-x. (2013). Modeling and optimization of the multiobjective stochastic joint replenishment and delivery problem under supply chain environment. *The Scientific World Journal*, 2013:1–11.

Whitin, S. (1953). *The Theory of Inventory Management*. Princeton University Press.

Williams, B. and Tokar, T. (2008). A review of inventory management research in major logistics journals: Themes and future directions. *The International Journal of Logistics Management*, 19:212–232.

Zhao, Q.-H., Wang, S.-Y., Lai, K.-K., and Xia, G.-P. (2004). Model and algorithm of an inventory problem with the consideration of transportation cost. *Computers & Industrial Engineering*, 46:389–397.

Paper 2

Multi-stage scenario generation by the combined moment matching and scenario reduction method

Multi-Stage Scenario Generation by The Combined Moment Matching and Scenario Reduction Method

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Abstract

We describe an opportunity to speed up multi-stage scenario generation and reduction using a combination of two well known methods: the moment matching method (Høyland and Wallace 2001) and the method for scenario reduction to approximately minimize a metric (Heitsch and Römish 2009). Our suggestions is to combine them rather than using them in serial by making use of a stage-wise approximation to the moment matching algorithm. Computational results show that combining the methods can bring significant benefits.

Keywords: Scenario Generation; Multi-stage stochastic programming; moment matching; scenario reduction

1 Introduction

Multi-stage, stochastic, optimization models receive increased interest as solver technology improves. Many solvers require that stochastic data be presented in the form of discrete realizations with attached probabilities. In the multi-stage case, they are almost always organized into a tree with the property that scenarios with the same realization up to decision stage share a node at that stage. See, e.g., (King and Wallace, 2010) for more discussion of scenario trees and stochastic optimization modeling.

In this note we describe an opportunity to speed up multi-stage scenario generation and reduction using a combination of two well known methods: the moment matching (MM) method (Høyland and Wallace, 2001) and the method for scenario reduction to approximately minimize a metric (SR) (Heitsch and Römisch, 2009). While the MM is designed to generate scenarios, SR is used to reduce an existing scenario tree to a tractable size. Although Monte Carlo methods can be used to generate a scenario tree before reducing it (Geyer et al., 2013; Hochreiter and Pflug, 2007; Latorre et al., 2007), a sensible alternative is to use MM to generate scenarios and then SR to make the scenario tree tractable (see (Feng and Ryan, 2013)). In all cases, the scenario tree is constructed in full before reduction. Our suggestions is to combine MM and SR rather than using them in serial. This is done making use of a stage-wise approximation to the MM algorithm. We give a brief review of the methods in the next two sections and then give the combined method in Section 4. Experimental results that confirm significant speed improvements are presented in Section 5, along with concluding remarks.

2 The Moment Matching Method for Scenario Generation

The idea of MM is to match the statistical properties of the generated scenarios with those of the observed data process. Following the notation presented by Høyland and Wallace (2001), define Γ as a set of statistical properties to be matched, and Γ_{VALi} as the observed value of statistical property i from Γ. Then let N be the number of random variables, T be the number of stages and Θ_t be the number of conditional outcomes in stage t. Define the outcome vector x of dimension $N \cdot \Theta_1 + N \cdot \Theta_1 \cdot \Theta_2 + ... + N \cdot \Theta_1 \cdot \Theta_2 \cdot \cdots \Theta_T$, which means that there are $\Theta_1 \cdot \Theta_2 \cdot \cdots \Theta_t$ outcomes of each variable $n = \{1, ..., N\}$ in stage $t = \{1, ..., T\}$. The probability vector ρ of dimension $\Theta_1 + \Theta_1 \cdot \Theta_2 + ... + \Theta_1 \cdot \Theta_2 \cdots \Theta_T$. The function $f^i(x,\rho)$ is the mathematical expression for statistical property i in Γ. Finally, let w_i be the weight for statistical property i in Γ.

We then construct vectors x and ρ by solving the non-linear optimization problem:

$$\min_{x,\rho} \sum_{i \in \Gamma} w_i \left(f^i(x,\rho) - \Gamma_{VALi} \right)^2 \tag{1}$$

$$\rho_{1,1} = 1 \tag{2}$$

$$\sum_{k=(j-1)\Theta_t+1}^{j\Theta_t} \rho_{t,k} = 1, \ \forall t = 2,...,T, j = 1, ..., \prod_{h=1}^{t-1} \Theta_h \tag{3}$$

$$\rho_{t,k} > 0, \ \forall t \in T, \ k \in 1, ..., \prod_{h=1}^{t} \Theta_h \tag{4}$$

In this model, $\rho_{t,k}$ expresses probability of the outcome $k = 1, ..., \prod_{h=1}^{t} \Theta_h$ in stage $t = 1, ..., T$.

In principle, one can use as many moments and state dependent statistical properties as desired. To be concrete, we will refer to an example where we are matching the three first moments of demand for a set of products $\mathcal{N}$ and the paired correlation function as statistical properties in the objective function, and where the mean value is the only state dependent statistical property. The data set used to compute the target values of the statistical properties was taken from the Norwegian company Stokke, and includes data on the demand of a set of products for children.

The statistical properties can be expressed in the following way:

$$f^{mean}_{n,t,k}(x,\rho) = \sum_{g=(k-1)\Theta_t+1}^{k\Theta_t} x_{n,t,g}\rho_{t,g}, \ \forall n \in \mathcal{N}, \ t = 2,...,T, \ k = 1,..., \prod_{h=1}^{t-1} \Theta_h \tag{5}$$

$$f^{variance}_{n,t,k}(x,\rho) = \frac{\Theta_t}{\Theta_t - 1} \sum_{g=(k-1)\Theta_t+1}^{k\Theta_t} \left(\left(x_{n,t,g} - f^{mean}_{n,t,k}(x,\rho) \right)^2 \rho_{t,g} \right),$$
$$\forall n \in \mathcal{N}, \ t = 2,...,T, \ k = 1,..., \prod_{h=1}^{t-1} \Theta_h \tag{6}$$

$$f^{skewness}_{n,t,k}(x,\rho) = \frac{\sqrt{\Theta_t(\Theta_t - 1)}}{\Theta_t - 2} \frac{\sum_{g=(k-1)\Theta_t+1}^{k\Theta_t} \left(\left(x_{n,t,g} - f^{mean}_{n,t,k}(x,\rho) \right)^3 \rho_{t,g} \right)}{\left(\sum_{g=(k-1)\Theta_t+1}^{k\Theta_t} \left(\left(x_{n,t,g} - f^{mean}_{n,t,k}(x,\rho) \right)^2 \rho_{t,g} \right) \right)^{3/2}},$$
$$\forall n \in \mathcal{N}, \ t = 2,...,T, \ k = 1,..., \prod_{h=1}^{t-1} \Theta_h \tag{7}$$

$$f^{correlation}_{n,m,t,k}(x,\rho) =$$
$$\frac{\sum_{g=(k-1)\Theta_t+1}^{k\Theta_t} \left(\left(x_{n,t,g} - f^{mean}_{n,t,k}(x,\rho) \right) \left(x_{m,t,g} - f^{mean}_{n,t,k}(x,\rho) \right) \rho_{t,g} \right)}{\sqrt{\sum_{g=(k-1)\Theta_t+1}^{k\Theta_t} \left(\left(x_{n,t,g} - f^{mean}_{n,t,k}(x,\rho) \right)^2 \rho_{t,g} \right) \sum_{g=(k-1)\Theta_t+1}^{k\Theta_t} \left(\left(x_{m,t,g} - f^{mean}_{m,t,k}(x,\rho) \right)^2 \rho_{t,g} \right)}},$$
$$\forall n,m \in \mathcal{N}, \ t = 2,...,T, \ k = 1,..., \prod_{h=1}^{t-1} \Theta_h \tag{8}$$

In formulas (5-8) $x_{n,t,g}$ expresses the value of variable x for product $n \in \mathcal{N}$ in outcome $g = 1,..., \prod_{h=1}^{t} \Theta_h$ on stage $t \in T$.

In addition to matching the first three moments and the correlation matrix, Høyland and Wallace (2001) suggest using autocorrelation in a special way. Let $\hat{i}$ be the index for the mean, so $\Gamma_{VAL\hat{i}}$ is treated as the observed value for the mean. For a particular scenario tree parent with indexes $(n,t\text{-}1,k)$, $\forall n \in \mathcal{N}; \ t = 1,...,T; \ k = 1,.., \prod_{h=1}^{t} \Theta_h$ the computation of $\Gamma_{VAL\hat{i}}$ for its children is as follows:

$$\alpha_n \cdot x_{n,t\text{-}1,k} + (1 - \alpha_n) \cdot \mu_n, \tag{9}$$

where μ_n represents the grand mean demand for product $n \in \mathcal{N}$ and $\alpha_n \in [0,1]$ represents the mean reversion factor for product $n \in \mathcal{N}$. The expected demand on the first stage can be defined in two different ways. The first option is to set it equal to μ_n. The second option is to specify the current actual value of demand and use the formula above to calculate the expected demand on the first stage. Note that it is the presence of state dependent factors such as mean reversion makes stage-wise decomposition an approximation. Such factors, and mean reversion in particular, can be very important in modeling multi-stage stochastic processes.

As pointed out by Høyland and Wallace (2001), in general such an optimization problem is not convex. So, one is likely to find a locally optimal solution, rather that one that is globally optimal. However, for purposes of scenario generation, finding a perfectly optimal match of properties is not necessary. Consequently, we propose decomposition of the presented problem into sub-problems, so that a matching problem can be solved for each stage separately. The stage-by-stage algorithm can be stated as follows:

Step 1. Set $t = 1$ and $\Gamma_{VAL\hat{i}} = \mu_n$ for each $x_{n,t,1}, \ \forall n \in \mathcal{N}$

Step 2. Set $t = t + 1$. For each parent $x_{n,t-1,k}, \ \forall n \in \mathcal{N}, \ k = 1, ..., \prod_{h=1}^{t-1} \Theta_h$ compute $\Gamma_{VAL\hat{i}}$ for its children using formula (9).

Step 3. Solve the moment matching optimization model (1-4) with respect to the t-th stage components only. If $t < T$, go to step 2.

Step 4. Construct scenarios $\mathcal{S}$, where $\tilde{x}_t^i, \ \forall i \in \mathcal{S}$ is a vector of n variables related to scenario i at stage t, and ρ_t^i is the conditional probability of obtaining vector $\tilde{x}_t^i$. Then $\rho^i = \prod_{t=1}^{T} \rho_t^i$ is the probability of a scenario $i \in \mathcal{S}$.

We have compared the performance of the original MM with the performance of the decomposed MM based on five instances of each of two configurations types. The results are shown in Table 1). The configurations are labeled (N, T, b) where b is the branching factor, indicating the constant number of child nodes generated from each node at stage $t = 1, ..., T - 1$. Both algorithms were coded using Python and Pyomo (Hart et al., 2011); the nonlinear minimization problems were solved using the IPOPT (Wächter and Biegler, 2006) nonlinear solver version 3.10.2. The CPU running times were computed for an 8x Intel(R) Core(TM) i7-2600 CPU 3400 GHz processor with 15 GB RAM. The same arbitrary initial solution was used as a starting point for both algorithms.

Table 1: The Original and the Decomposed MM: Comparison of Performance based on averages taken over five replicates for each configuration.

Configuration	Original MM		Decomposed MM	
	CPU (sec)	Objective	CPU (sec)	Objective
(2,4,6)	333.84	0.317	11.15	$1.67 \cdot 10^{-6}$
(2,5,6)	6942.10	$1.97 \cdot 10^{13}$	135.49	0.571

One can see that the original MM algorithm requires more time to find a solution and the quality of this solution is actually much worse then quality of the solution found by the decomposed MM for these instances. This is because the original problems are so large. If we use the solution obtained by the decomposed MM as a starting point for the original MM, it was improved in approximately 10% of the cases.

Nevertheless, the average objective values do not present the entire picture of algorithms' performance. In the case of the (2,4,6) configuration, the original MM got an objective value significantly different from 0 (1.5844) only once out of five instances. That resulted in a comparably high average objective value. For the same instance the decomposed MM resulted in an objective value of factor 10^{-6}. In the case of the (2,5,6) configuration, when the original MM was used, the objective function for one of the instances was equal to 5.21, which is comparably close to 0. However, it didn't affect the average objective value, since the smallest objective value among other instances was of factor 10^{6}. In the same cases, when the decomposed MM was used the highest objective value was equal to 1.44.

We also conducted some experiments with configuration (44,4,12). IPOPT could not converge on a solution for the original MM for this instance, but it was able to for the decomposed MM (it took 10739.69 CPU seconds). The reason is in the size of the optimization problem to be solved. In the case of the original MM, instances of configuration (44,4,12) are characterized by one non-linear problem with a highly non-linear objective and 84825 variables. The decomposed MM for the same instance is characterized by 157 nonlinear problems with 540 variables each.

It is clear *a priori* that it is computationally expedient to decompose by stages. An important conclusion from these experiments is that for practical reasons, a decomposition by stages may be required in some situations.

3 The Scenario Reduction Method

When the number of scenarios generated is large, one could be interested in a scenario tree reduction in order to be able to run the stochastic optimization model in a shorter amount of time. The scenario reduction (SR) method proposed by Heitsch and Römisch (2009) and the *forward construction* algorithms based on this method (Eichhorn et al., 2010) are intended to accomplish this task. The forward construction algorithm successively computes partitions of scenario set $\mathcal{S}$ into $\lambda_t,\ t = 1, ..., T$ clusters of the form:

$$\Delta_t := \left\{\Delta_t^1, ..., \Delta_t^{\lambda_t}\right\}, \quad \lambda_t \in \mathbb{N}, \tag{10}$$

where $\mathbb{N}$ is any natural number not exceeding Θ_t.

The elements of Δ_t are called clusters. Now let us consider the algorithm:

Step 1. Define $\Delta_1 = \mathcal{S}$ and set $t := 2$

Step 2. For each cluster of scenarios Δ_{t-1}^i, run the scenario reduction procedure with respect to $\tilde{x}_t^j,\ j \in \Delta_{t-1}^i$.

Step 3. Obtain the mapping β_t^λ from the deleted scenarios $\mathcal{J}_t^\lambda$ to the remaining scenarios $\mathcal{S}_t^\lambda$ such that:

$$\beta_t^\lambda(i) \in \underset{j \in \mathcal{S}_t^\lambda}{\operatorname{argmin}} \|\tilde{x}_t^i - \tilde{x}_t^j\|,\ i \in \mathcal{J}_t^\lambda \tag{11}$$

Step 4. Define the overall mapping α_t from the original set of scenarios $\mathcal{S}$ to the new set of scenarios $\mathcal{S}$:

$$\alpha_t(i) = \begin{cases} \beta_t^{\lambda}(i), \ i \in \mathcal{J}_t^{\lambda}, \ \textit{for some } \lambda = 1, ..., \lambda_{t-1} \\ i, \ otherwise \end{cases} \tag{12}$$

Then a new partition at t is:

$$\Delta_t := \left\{ \alpha_t^{-1}(i) \mid i \in \mathcal{S}_t^{\lambda}, \ \lambda = 1, ..., \lambda_{t-1} \right\} \tag{13}$$

If $t < T$, set $t := t + 1$ and continue from Step 2, otherwise go to Step 5.

Step 5. Define new set of scenarios according to the partition set Δ_T and mappings (12). Each scenario will be the centroid of a cluster. The probability of each scenario from a new set will then be equal to the sum of probabilities of the scenarios belonging to the same cluster, plus the probability of the scenario itself.

In Step 1 a single cluster consisting from all the initial scenarios is defined. In Steps 2-4 the number of scenarios is iteratively reduced. We consider all scenarios at a given stage and refer to a component of a scenario corresponding to a given stage as a *node*. Thus these steps can be understood as the selection of a given number of nodes λ_t from nodes belonging to the scenarios remaining in the previous stage. For example, consider stage 3. Suppose, that in the previous stage we have defined 2 clusters: Δ_2^1 and Δ_2^2. This means that only scenarios having the same node as the centroid scenarios on the second stage were selected for further consideration. Suppose also that $\lambda_3 = 2$. Then from all the third stage nodes belonging to the remaining scenarios of a given cluster Δ_2^{λ} we form two clusters on stage 3. So, in total we will have four clusters of scenarios after stage 3. Each of these clusters will have centroid scenarios with a given node at stage 3. Only scenarios which are going through these centroid nodes are selected for further consideration on the next stage.

Once the scenarios at all of the stages have been considered, the new set of scenarios from the remaining list of scenarios in the last stage are formed. To understand the procedure for selecting the centroid nodes of clusters in Step 2 and the mapping in Steps 3-4 of the forward construction algorithm, consider a set of nodes $\mathcal{I}_t$ with given coordinates $\tilde{x}_t^i, \ i \in \mathcal{I}$ and probabilities ρ_t^i. Selecting λ_t centroid nodes from them can be considered as a P-median problem, where the distance between the nodes i and j is defined as a norm of the difference between coordinates of the nodes $d_{i,j} = \|\tilde{x}_t^i - \tilde{x}_t^j\|$ and the weight of the node is denoted by probability of this node ρ_t^i.

The P-median problem is known to be $\mathcal{NP}$-hard, thus Eichhorn et al. (2010) proposed use of a greedy forward selection algorithm:

Step 1. Set $\mathcal{J} := \mathcal{I}_t$

Step 2. Determine an index $l \in \mathcal{J}$ such that

$$l \in \operatorname*{argmin}_{u \in \mathcal{J}} \sum_{k \in \mathcal{J} \setminus \{u\}} \rho_k \min_{j \notin \mathcal{J} \setminus \{u\}} \left| x_t^k - x_t^j \right|, \tag{14}$$

and set $\mathcal{J} := \mathcal{J} \backslash \{l\}$. If the cardinality of $\mathcal{J}$ equals to desired number of clusters go to termination step. Otherwise continue with a further index selection step.

Step 3. Find a mapping from the original set $\mathcal{J}$ to a reduced set $\mathcal{J}$ by assigning each node that is not in the new $\mathcal{J}$ to the closest median.

Instead of applying the forward selection heuristic one can try to solve a MIP formulation of the P-median problem, using a commercially available solver, which gives a provably bounded, nearly exact solution in comparable time for the problem we use as an example. Define two sets of variables: $\phi_i \in \{0, 1\} \ \forall i \in \mathcal{I}_t$ which indicates if scenario i was selected as centroid of a cluster on stage t ($\phi_i = 1$), and $\tau_{i,j} \in \{0, 1\} \ \forall i \in \mathcal{I}_t, \ j \in \mathcal{I}_t$ which denotes if scenario j is mapped to scenario i on stage t. Then the linear model for the P-median problem is:

$$\min \sum_{i \in \mathcal{I}_t} \sum_{j \in \mathcal{I}_t} \tau_{i,j} d_{i,j} \rho_t^j \tag{15}$$

subject to:

$$\sum_{i \in \mathcal{I}_t} \phi_i = \lambda_t \tag{16}$$

$$\sum_{i \in \mathcal{I}_t} \tau_{i,j} = 1, \ \forall j \in \mathcal{I}_t \tag{17}$$

$$\sum_{j \in \mathcal{I}_t} \tau_{i,j} \leq \phi_i \left(|\mathcal{I}_t| - 1 \right), \ \forall i \in \mathcal{I}_t \tag{18}$$

$$\tau_{i,j} \in \{0, 1\}, \ \forall i \in \mathcal{I}_t, \ j \in \mathcal{I}_t \tag{19}$$

$$\phi_i \in \{0, 1\}, \ \forall i \in \mathcal{I}_t \tag{20}$$

Applying the forward construction algorithm along with any algorithm for P-median problem will allow us to reduce the original set of scenarios to the desired size.

4 Combined Moment Matching Scenario Reduction method

It is easy to see that the steps of the stage-wise MM algorithm given in Section 2 can be combined with the steps of the forward construction algorithm for the SR method in two different ways. One way is to apply the MM algorithm and then apply the forward construction. Another way is to apply the two methods simultaneously while constructing the scenario tree stage by stage. The combined algorithm can significantly reduce the time required to construct the scenario tree.

The combined moment matching forward construction algorithm for the combined moment matching scenario reduction method is as follows:

Step 1. Set $t = 1$ and $\Gamma_{VAL\hat{i}} = \mu_n$ for each $x_{n,t,1}, \ \forall n \in \mathcal{N}$

Step 2. Set $t = 2$. Define set $\mathcal{K}_{t\text{-}1} = \left\{1, ..., \prod_{h=1}^{t\text{-}1} \Theta_h\right\}$

Step 3. Compute $\Gamma_{VAL\hat{i}}$ for the children of each parent $x_{n,t\text{-}1,k}$, $\forall n \in \mathcal{N},\ k \in \mathcal{K}_{t\text{-}1}$ using formula (9).

Step 4. Solve the moment matching optimization model (1-4) with respect to the t-th stage components only. Create nodes with variable vector $\tilde{x}_t^i = (x_{1,t,i}, ..., x_{n,t,i})^T,\ i = (k\text{-}1) \cdot \Theta_t, ..., k \cdot \Theta_t,\ \forall k \in \mathcal{K}_{t\text{-}1}$ and probabilities $p_{t,i}$.

Step 5. Create λ_t clusters of nodes with centroid nodes $x_t^j,\ j \in \{1, ..., \lambda_t\}$ by solving the P-median problem for the nodes at stage t. Set $\mathcal{K}_t = \{1, ..., \lambda_t\}$. If $t < T$, set $t := t + 1$ and continue from Step 3, otherwise go to Step 6.

Step 6. Construct scenarios $\mathcal{S}$ from obtained centroid nodes.

In other words, at each stage except the first one, we create a set of nodes using the Moment Matching method, then we reduce the number of nodes using the Scenario Reduction method. To create the nodes at the next stage we branch only from the remaining nodes using the given branching factor.

5 Experiments and Conclusions

Let us consider the process of scenario creation by two separated methods and the combined method. If the MM and SR methods are applied sequentially, then we will have to solve $1 + 1 \cdot \Theta_2 + ... + 1 \cdot \Theta_2 \cdot \cdots \Theta_{T\text{-}1}$ matching problems and $1 + 1 \cdot \lambda_2 + ... + 1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_{T-1}$ P-Median problems. If the two methods are combined, the total number of matching problems and the total number of P-median problems to solve will be the same and equal to $1 + 1 \cdot \lambda_2 + ... + 1 \cdot \lambda_2 \cdot \cdots \cdot \lambda_{T-1}$. Hence, the larger the difference between the branching factor Θ_t and the number of clusters λ_t, the greater time savings can be achieved.

Consider the following example: Suppose, that we have 4 stages in our scenario tree, the branching factor is chosen to be $\Theta_t = 12,\ \forall t = \{2, 3, 4\}$ and the number of clusters is $\lambda_t = 2,\ \forall t = \{2, 3, 4\}$. If the two methods are applied in series we will have to solve 157 nonlinear matching problems and 7 P-median problems. If we apply the combined Moment Matching Scenario Reduction method we will have to solve only 7 matching problems and the same number of P-median problems. It means that we will avoid solving 150 nonlinear matching problems.

Examples of time savings in seconds can be seen in Table 2. The configurations are labeled as (N, T, b), where b is the branching factor, indicating the constant number of child nodes generated from each node at stage $t = 1, ..., T - 1$. We formulated the models using Pyomo and solved them using IPOPT as a nonlinear solver and we used a greedy forward selection algorithm coded in Python for the P-median problems. The MM algorithm used in serial experiments (and in the combined method experiments) was decomposed by stages. For the simultaneous solution of all stages, the computational times are dramatically higher.

Table 2: CPU Time (seconds) for the Serial and Combined Moment Matching Scenario Reduction method

Instance Configuration	Serial MM	Serial SR	Combined MM and SR
(2,3,6)	0.54	0.12	0.36
(2,4,6)	1.73	0.13	0.57
(2,4,12)	8.89	0.15	0.92
(2,5,12)	104.52	0.29	1.69
(2,6,12)	3398.81	1.43	2.99

As we can see, whenever scenario generation may constitute a large and time consuming part of solving a stochastic optimization problem, using the combined moment matching scenario reduction method can bring significant benefits.

References

Eichhorn, A., Heitsch, H., and Römisch, W. (2010). Stochastic optimization of electricity potfolios: Scenario tree modeling and risk management. In *Handbook of Power Systems II*, Energy Systems, pages 405–42. Springer.

Feng, Y. and Ryan, S. (2013). Scenario construction and reduction applied to stochastic power generation expansion planning. *Computers & Operations Research*, 40(1):9–23.

Geyer, A., Hanke, M., and Weissensteier, A. (2013). Scenario tree generation and multi-asset financial optimization problem. *Operations Research Letters*, 41(5):494–498.

Hart, W. E., Watson, J.-P., and Woodruff, D. (2011). Pyomo: Modeling and solving mathematical programs in Python. *Mathematical Programming Computation*, 3(3):219–260.

Heitsch, H. and Römisch, W. (2009). Scenario tree reduction for multistage stochastic programs. *Computational Management Science*, 6, No 2:117–133.

Hochreiter, R. and Pflug, G. (2007). Financial scenario generation for stochastic multi-stage decision processes as facility location problems. *Annals of Operations Research*, 152(1):257–272.

Høyland, K. and Wallace, S. (2001). Generating scenario trees for multistage decision problems. *Management Science*, 47, No.2:295–307.

King, A. and Wallace, S. (2010). *Modelling with Stochastic Programming*. Springer.

Latorre, J., Cerisola, S., and Ramos, A. (2007). Clustering algorithms for scenario tree generation: Application to natural hydro inflows. *European Journal of Operational Research*, 181(3):1339–1353.

Wächter, A. and Biegler, L. (2006). On the implementation of a primal-dual interior point filter line search algorithm for large-scale nonlinear programming. *Mathematical Programming*, 106:25–57.

Paper 3

A Stochastic Programming Approach to solve a Coordinated Capacitated Stochastic Dynamic Demand Lot-Sizing Problem with Emergency Supplies

A Stochastic Programming Approach to solve a Coordinated Capacitated Stochastic Dynamic Demand Lot-Sizing Problem with Emergency Supplies

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Abstract

We consider the coordinated capacitated lot-sizing problem (CCLSP) with dynamic, stochastic demand, previously not addressed in the literature. We provide important extensions to the classical version of related problems in order to provide solutions that reflect conditions in a real-world application. To test the advantage of the stochastic model with respect to a deterministic model we applied the production plan in a rolling horizon settings with data from a company providing worldwide distribution of children's furniture and equipment. The stochastic model out-performed the deterministic model with 51.4% lower total cost, leading to a significant reduction in inventory costs and to a slight increase in use of emergency supplies.

Keywords: Inventory; Stochastic programming; Coordinated capacitated lot-sizing problem; Stochastic dynamic problem; Scenario generation

1 Introduction

In this paper we consider the coordinated capacitated lot-sizing problem (CCLSP) with dynamic, stochastic demand. Though the deterministic version of the problem has been well studied (see, e.g., Robinson et al. (2009)), the CCLSP with stochastic demand has not been addressed in the literature before. The main difference between the coordinated capacitated lot-sizing problem and the stochastic capacitated lot-sizing problem (SCLSP) is the presence of joint setup costs. We apply a stochastic programming approach, previously used by Brandimarte (2006) to address the SCLSP, to solve a variant of the CCLSP taken from the real world.

The problem comes from the Norwegian company Stokke, designing and distributing products for children. The company operates a supply chain, which is characterized by lead times significantly exceeding the length of the replenishment period. In practice this means that there are several (normally three) orders on the way from a supplier to the company's warehouse at any time. Hence, when placing an order one shall account for at least three periods ahead to optimize the overall performance of the system.

The problem considered in this paper is related to the class of lot-sizing problems (LSP). Harris (1913a,b) introduced the first model of this class, assuming single-item static deterministic demand, continuous time and unlimited lot-size. Since then, a vast variety of lot-sizing problems have been discussed in the literature. Robinson et al. (2009) presented the most recent review of the literature on the topic. Authors have mostly concentrated on the discussion of the coordinated deterministic dynamic demand LSP. This problem is characterized by multiple items, which can be jointly replenished on a single link. The demand for each of the items is deterministic, but dynamic. There are two main variants of the problem: capacitated and uncapacitated. Different problem formulations are presented in the literature together with solution approaches.

The natural extension of the coordinated capacitated deterministic demand LSP would be a problem with stochastic demands. Robinson et al. (2009) considered the Joint Replenishment Problem (JRP) as a sub-class of the LSP. Khouja and Goyal (2008) presented the most recent review of the literature on the JRP. The paper includes a discussion of the JRP under stochastic demands among others. Unlimited replenishment size and stationary demands are assumed for this kind of problem. Lee and Chew (2005) developed an algorithm for a version of the problem with independent and auto-correlated demands. Rubasheuski et al. (2014a) used a JRP setting to consider a problem with stationary stochastic demands and modified setup costs.

While considering stochastic demands in more general settings of LSP, most authors concentrate on single-item models (Levi and Shi, 2013; Levi et al., 2005; Smith and Tan, 2013; Delaert and Melo, 1998). Only a few authors aimed to solve capacitated stochastic demand LSP. Tempelmeier (2011) used a column generation heuristic to obtain the fixed production plan for the whole time horizon. Helber et al. (2013) proposed two different procedures to approximate the stochastic nature of demand and then used a fix-and-optimize heuristic in order to find a solution for the problem.

Another area of research closely related to the coordinated LSP deals with the stochastic economic lot-scheduling problem (SELSP). While solving a SELSP one aims to establish a set of rules (policy) to manage a production process, rather than a concrete production plan. Together with absence of joint setup costs in the classical model formulation, this makes the SELSP different from the CCLSP. The most recent review on this topic is given by Winands et al. (2011).

We propose to use a stochastic programming approach (Ruszczynski and Shapiro, 2003) in order to solve a special case of a coordinated capacitated stochastic dynamic demand LSP with emergency supplies. Some authors have used scenario trees in order to deal with stochastic demand LSP (Li and Thorstenson, 2013; Brandimarte, 2006; Beraldi et al., 2006). They developed algorithms competing to find optimal solutions to the classically formulated problems.

The current work differs from previous research in several ways: a new variant of a coordinated capacitated LSP is considered; stochastic, non-stationary in mean, correlated demands are assumed; back-order costs are considered as part of the total cost function; back-orders are limited by the target service level and cannot exceed a certain share of demand during a particular period; emergency supplies are allowed; the setup costs are dependent on the number and size of transportation units used.

First, a new variant of the coordinated capacitated LSP with emergency supplies is considered. The formulation of the problem is presented together with a solution approach. Second, the problem is solved for multiple items with stochastic, non-stationary in mean, correlated demands. Third, the proposed problem formulation was applied to solve the real world case of Stokke AS, indicating significant improvements compared to the common business practice.

The rest of the paper is organized as follows. Section 2 is devoted to the description of assumptions and problem formulation. Section 3 is used to discuss the scenario generation procedure and the solution approach. Section 4 presents the results of the algorithm implementation on a given problem together with the discussion of possible extensions of the model. Section 5 summarizes the findings and presents conclusions.

2 Problem statement

We assume a system where multiple products are ordered from a single source of supply. The time between orders is fixed and orders are transported using transport units of different capacities. The lead time for all units, regardless of the capacity, is identical and constant. It is equal to *L* time intervals where *L* is an integer value. To illustrate this we make use of Figure 1. Assume that the lead time length *L*=3. Then order o_1, placed in the beginning of the first period, is delivered in the beginning of time period *t*=4. Product demands are stochastic, correlated between each other, and auto-correlated. So the expected demand in period *t*, can be expressed as follows:

$$E\left(D_t\right) = \alpha \cdot D_{t-1} + (1-\alpha) \cdot \mu, \quad (1)$$

where μ represent the grand mean demand and $\alpha \in [0,1]$ represents the mean reversion factor. Other statistical properties of demand are assumed to be state-independent.

Considering demand within one time interval, we assume that it arrives in a continuous manner at a constant rate. Then there could be three possible inventory states in the end of the period as shown on Figure 2:

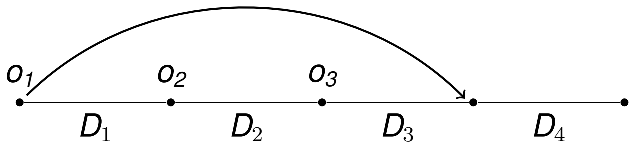

Figure 1: An example of order placing and allocation at a warehouse for lead time length L=3 time periods

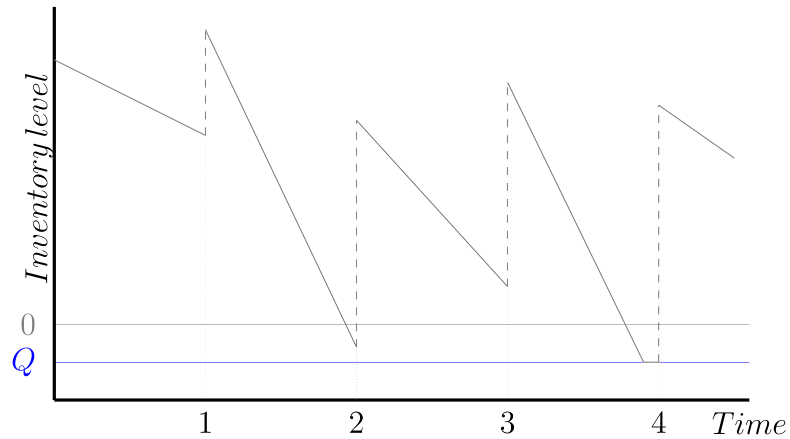

Figure 2: Inventory level fluctuations

- The demand does not exceed the available inventory, as in period 1 and 3. Then neither backorder nor emergency supply will occur.
- The demand exceeds the available inventory, but the limit for allowed backorders, *Q*, is not reached (period 2), then no emergency supply will occur.
- The demand exceeds the available inventory and the limit for allowed backorder quantity, *Q*, is reached (period 4), then the option of emergency supply is used.

We assume that the time to deliver an item using the emergency supply source is negligible, and that the cost of delivering one unit of the product is constant and exceeds the backorder cost per unit of this product. Hence, the emergency supply is used only when the limit for the number of backordered units is reached.

Table 1: Notation

Sets	
$\mathcal{N}$	= set of products
$\mathcal{M}$	= set of transport unit sizes
Parameters	
T	= number of time periods to review
L	= length of the lead time as number of time periods, integer
$D_{n,t}$	= demand for product n in period t, $n \in \mathcal{N}$, $t \in \{1, .., T+L+1\}$
$O_{n,t}$	= order for product n which was placed at time t, $n \in \mathcal{N}$, $t \in \{2\text{-}L, .., 0\}$
V_n	= volume per unit of product n, $n \in \mathcal{N}$
U_m	= capacity of the transport unit of size index m, $m \in \mathcal{M}$
E_m	= transportation cost per transport unit of size index m, $m \in \mathcal{M}$

Table 1: (continued)

F	= transportation cost per cubic meter from emergency supply source
W_n	= unit cost for product n, $n \in \mathcal{N}$
H	= one period holding cost as a fraction of unit cost
B	= one period backorder cost as a fraction of unit cost
Q_n	= limit for backorder quantity of product n, $n \in \mathcal{N}$
I_n	= initial inventory of product n, $n \in \mathcal{N}$

Variables	
$o_{n,t}$	= number of units of product n ordered at the beginning of period t, $n \in \mathcal{N},\ t \in \{1, .., T\}$
$y_{m,t}$	= number of transport units of size index m used for orders placed at the beginning of period t, $m \in \mathcal{M},\ t \in \{1, .., T\}$
$i_{n,t}$	= inventory of product n at the end of period t, $n \in \mathcal{N},\ t \in \{L, .., T + L\}$
$q_{n,t}$	= backorder quantity of product n at the end of period t, $n \in \mathcal{N},\ t \in \{L, .., T+L\}$
$f_{n,t}$	= quantity of product n supplied during period t using the emergency source, $n \in \mathcal{N},\ t \in \{L, .., T+L\}$

Given these assumptions and with use of parameters and variables shown in Table 1 , the average inventory of product n during period t can be expressed in the following way:

$$\begin{cases} \left(i_{n,t-1} + o_{n,t} - q_{n,t-1}\right)^2 / 2 \left(i_{n,t-1} + o_{n,t} - q_{n,t-1} + q_{n,t} + f_{n,t}\right), & if\ i_{n,t} = 0, \\ \frac{1}{2}\left(i_{n,t-1} + o_{n,t} - q_{n,t-1} + i_{n,t}\right), & otherwise. \end{cases} \tag{2}$$

In order to simplify the calculations of the objective function we assume that the average inventory of product n can be always expressed as:

$$\frac{i_{n,t-1} + i_{n,t} + o_{n,t} - q_{n,t-1}}{2} \tag{3}$$

Joint setup costs and item specific setup costs are replaced with transportation costs which depends on the number of transport units of given capacities being ordered. Making use of parameters and variables shown in Table 1 one can express the transportation (setup) costs as follows:

$$Transportation\ costs = \sum_{t=1}^{T} \sum_{m \in \mathcal{M}} y_{m,t} \cdot E_m \tag{4}$$

A model for a deterministic version of the problem is formulated in Table 2 with use of parameters and variables shown in Table 1 and an extra set of assumptions:

- Product demands can be determined in advance with 100% confidence for T time periods.

- Inventory and transportation costs are fixed for all periods.
- An order is placed in the beginning of the time period.
- Product demands are continuous and arrive with a constant rate.
- Inventory level or backorder quantity are measured in the end of the time period.

Table 2: Deterministic multi period model

$$min\,C = \sum_{t=1}^{T} \sum_{m \in \mathcal{M}} y_{m,t} \cdot E_m + \sum_{t=L+1}^{T+L} \sum_{n \in \mathcal{N}} \frac{i_{n,t-1} - q_{n,t-1} + o_{n,t-L} + i_{n,t}}{2} \cdot W_n \cdot H + \sum_{t=L+1}^{T+L+1} \sum_{n \in \mathcal{N}} q_{n,t} \cdot W_n \cdot B + \sum_{t=L}^{T+L+1} \sum_{n \in \mathcal{N}} f_{n,t} \cdot F \cdot V_n \quad (5)$$

Subject to:

$$i_{n,L} - q_{n,L} - f_{n,L} = \sum_{t=1-L}^{0} O_{n,t} + \sum_{t=1}^{L} D_{n,t} + I_n, \ \forall n \in \mathcal{N} \quad (6)$$

$$o_{n,t} + i_{n,t+L-1} + q_{n,t+L} + f_{n,t+L} = D_{n,t+L} + i_{n,t+L} + q_{n,t+L-1}, \quad \forall n \in \mathcal{N},\ t \in \{1, .., T\} \quad (7)$$

$$\sum_{n \in \mathcal{N}} o_{n,t} \cdot V_n \leq \sum_{m \in \mathcal{M}} y_{m,t} \cdot U_m, \ \forall t \in \{1, .., T\} \quad (8)$$

$$o_{n,t} \geq 0, \forall\, n \in \mathcal{N},\ t \in \{1, .., T\} \quad (9)$$

$$q_{n,t} \leq Q_n, \forall\, n \in \mathcal{N},\ t \in \{L, .., T+L\} \quad (10)$$

$$q_{n,t} \geq 0,\ i_{n,t} \geq 0, \forall\, n \in \mathcal{N},\ t \in \{L, .., T+L\} \quad (11)$$

$$y_m \in \mathbb{N}, \forall\, m \in \mathcal{M} \quad (12)$$

In the deterministic case we consider a multi-period problem where the objective (5) is to minimize the sum of transportation and inventory holding costs. Estimating the inventory costs, we assume that demand is constant and continuous during the time period. Then the average inventory level during the period is measured as the sum of the inventories in the beginning and end of the period divided by two. The inventory level in the beginning of a time period is equal to the sum of an order arriving in the beginning of this time period and the inventory (or backorder) left at the end of the previous time period.

It should be noted, that such an estimation of the inventory costs leads to an overestimation of the real costs. In our model, it is assumed that the inventory level can drop to 0 only in the end of the time period, however in practice it can get down to 0 in the middle of the period. Hence the estimated average inventory level exceeds the real average inventory level. This leads to an overestimation of the inventory holding costs. Such a drawback can be eliminated by a significant complication of the model, which we consider unnecessary, as the inventory level in practice almost never drops down to 0.

As all the demands are assumed to be known in advance, the orders for all review periods *T* can be optimized simultaneously at one stage. But even though we know the demands exactly, we allow having inventory, backorder and emergency supplies to make a better use of fixed transport unit capacities.

Constraint set (6) estimates the inventory and backorder amount at the end of time period *L*, i.e., just before the arrival of the first order. As both the inventory level and the backorder quantity affect the objective function, at least one of them will be equal to 0 in an optimal solution. Constraint set (7) requires that demand for each product was balanced with available inventory during each time period.

The constraint set (8) requires that all orders fit into the ordered transport capacity. Constraint sets (9-11) set upper and lower bounds for the number of ordered products, inventory level and backorder level values for each of the products and each of the relevant periods. Constraint set (12) requires that the number of transport units of each type and during each period are non-negative integers.

3 Multi-stage Stochastic model formulation

In the real world demands are almost never known in advance. Thus we suggest that the problem discussed in this paper can be better handled using a stochastic programming approach. In real life one typically makes a decision only about the first order, while decisions about the next orders will be made at the time they are placed. We propose the Multi-Stage Stochastic Model presented in Table 3 to account for these changes.

Notation for the Multi-Stage Stochastic Model will stay the same as in the previous section, but decisions will be made at *T+1* stages. At each stage $t \in \{1,..,T\}$ we will make a decision about the order size $o_{n,t}$ for each product *n*, so the stages will be consistent with time periods. At the last stage *T+1* we will make "decisions" about all the inventory, backorder and emergency order quantities for periods $t \in \{L+1,..,T+L+1\}$, where L is the length of the lead time. This means that the last stage is not consistent with one single

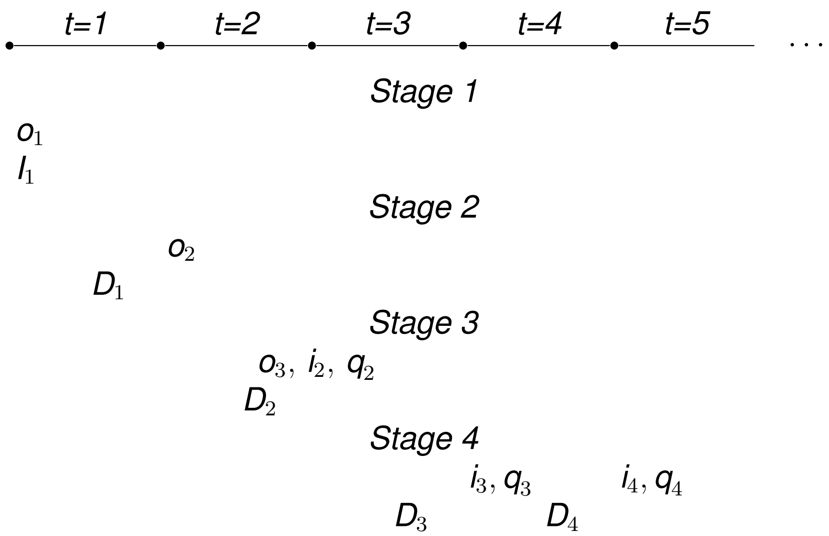

Figure 3: Relationship between stages and time periods in the Multi-Stage Stochastic Model

time period. One can make use of Figure 3 to better understand the relationship between time periods and stages.

To simplify the modeling task we assume that demands for each product during each period can be described using a discrete distribution with a limited number of options. Then using any scenario generation method, we can get a set of possible scenarios $\mathcal{S}$.

The solution system of a multi-stage model must be *admissible* and *implementable*. A solutions system, which satisfies the constraints for all of the scenarios, is referred to as *admissible*. We refer to a system of solutions as *implementable* if, for scenarios *s* and *s'* that follow the same branch of the scenario tree up to stage *k*, it is true that $o_n(s, k') = o_n(s', k')$ for all $1 \leq k' \leq k$ and each *n* in $\mathcal{N}$. The set of all solution systems for a given set of scenarios $\mathcal{S}$ is referred as $\mathcal{D}_{\mathcal{S}}$. Shorthand *X* is referred to as the entire solution system of *x* vectors (that is, *X*=*x*(1,1), ... ,*x*($\mathcal{S}$,*K*), where *K* is the last considered stage).

Table 3: Multi-Stage Stochastic Model

$$min\, C = \sum_{s \in \mathcal{S}} p_s \cdot \Bigg(\sum_{t=1}^{T} \sum_{m \in \mathcal{M}} y_{m,t,s} \cdot E_m + \sum_{t=L+1}^{T+L} \sum_{n \in \mathcal{N}} \frac{i_{n,t\text{-}1,s} - q_{n,t\text{-}1,s} + o_{n,t\text{-}L,s} + i_{n,t,s}}{2} \cdot W_n \cdot H + \sum_{t=L+1}^{T+L} \sum_{n \in \mathcal{N}} q_{n,t,s} \cdot W_n \cdot B + \sum_{t=L}^{T+L} \sum_{n \in \mathcal{N}} f_{n,t,s} \cdot V_n \cdot F \Bigg) \quad (13)$$

Subject to:

Table 3: (continued)

$$i_{n,L,s} - q_{n,L,s} - f_{n,L,s} = \sum_{t=1\text{-}L}^{0} O_{n,t} + \sum_{t=1}^{L} D_{n,t,s} + I_n,\ \forall n \in \mathcal{N}, s \in \mathcal{S} \quad (14)$$

$$o_{n,t,s} + i_{n,t+L\text{-}1,s} + q_{n,t+L,s} + f_{n,t+L,s} = D_{n,t+L,s} + i_{n,t+L,s} + q_{n,t+L\text{-}1,s},$$
$$\forall n \in \mathcal{N},\ t \in \{1,..,T\},\ s \in \mathcal{S} \quad (15)$$

$$\sum_{n \in \mathcal{N}} o_{n,t,s} \cdot V_n \leq \sum_{m \in \mathcal{M}} y_{m,t,s} \cdot U,\ \forall s \in \mathcal{S},\ t \in \{1,..,T\} \quad (16)$$

$$X \in \mathcal{D}_{\mathcal{S}} \quad (17)$$

$$o_{n,t,s} \geq 0, \forall n \in \mathcal{N},\ t \in \{1,..,T\},\ s \in \mathcal{S} \quad (18)$$

$$q_{n,t,s} \leq Q_n,\ \forall n \in \mathcal{N},\ t \in \{L,..,T+L\},\ s \in \mathcal{S} \quad (19)$$

$$i_{n,t,s} \geq 0, q_{n,t,s} \geq 0,\ \forall n \in \mathcal{N},\ t \in \{L,..,T+L\},\ s \in \mathcal{S} \quad (20)$$

$$y_{m,t,s} \in \mathbb{N},\ \forall m \in \mathcal{M},\ t \in \{1,..,T\},\ s \in \mathcal{S} \quad (21)$$

The objective function (13) consists of *T+1* stage costs. On each stage $t \in \{1,..,T\}$ costs are associated with expenses on transport needed to deliver the order placed on this stage. As the ordering decisions are taken on stage *t*, the transportation costs are also associated with the corresponding stage. The costs on stage *T+1* consist of inventory holding, backorder costs and emergency supply costs. Parameter p_s is the estimated probability of a given scenario *s*. Then the expected costs are dependent on the probability of each scenario and the costs associated with this scenario.

The set of constraints (14-16) correspond to the constraints (6-8) of the deterministic model . The only difference is that they are now valid for all of the scenarios.

Constraint set (17) requires that the system of solution vectors is *implementable*. This means that the first stage variables must have the same values for all of the scenarios.

The second stage variables will be the same for all branches of the scenario tree starting from the same second-stage node, etc. Assume that there can be six different outcomes of the stochastic process on the second stage, i.e., each of the scenarios goes through one of the six nodes on the second stage. Hence, all scenarios going through the same node, should have the same values of the decision variable on the second stage. Such logic applies for all the stages and scenarios.

Note that we are only interested in the solution elements corresponding to the first time period. At the start of the second and any other time period the model will be run again with updated demands and scenarios.

4 Scenario Tree Generation and Stability Testing

To run the Stochastic Programming Model we need a set of scenarios to review. In the real-world problem that motivates our work, products are typically produced in ensembles. For example, you can order a stroller, which includes shopping bag, parasol, cover coat etc. of the same color. The demand for all these separate items will be positively correlated but not identical (as not everyone orders a parasol, for example). At the same time the demand for strollers of different colors are likely to be negatively correlated. Indeed, the experience of Stokke shows that if the demand for green strollers goes up, the demand for red ones typically decreases. Hence demands are stochastic, auto-correlated and correlated both within and outside of the product families.

There are many ways to construct scenario trees in order to reflect the stochastic nature of the process (Kaut and Wallace, 2007; King et al., 2012). The combined moment matching and scenario reduction method (Rubasheuski et al., 2014b) is used to create scenario trees in this paper in order to reflect the properties of demand efficiently. Since demand for all the products is assumed to be random, we are matching its three first moments and the paired correlation function as statistical properties in the objective function. The mean value is the only state-dependent statistical property, while other properties are assumed to be stationary. At each time period $t = 2...T$ the number of possible outcomes from the parent node is reduced to 3. In our practical example the scenario tree will include 729 scenarios.

The methodology proposed by King et al. (2012) was used to test the in-sample and out-of-sample stability of the scenario generation procedure. Since the procedure to construct the scenarios is deterministic, the in-sample stability was tested using 10 scenario trees including from 725 to 734 scenarios. The standard variation of objective function value made up 0.0005% of the mean, indicating that the model is in-sample stable. The out-of-sample stability was tested fixing the root solution from the first scenario tree and solving the model for other scenario trees. Since the first stage decisions were exactly the same for all the scenario trees, there was no variation of objective function value, indicating that the model is out-of-sample stable. Thus we can claim that the solution does not depend on a specific scenario tree.

5 Model implementation and results

To evaluate the performance of the multi-stage stochastic model compared to the performance of the deterministic model, we used a data sample from Stokke, which included 44 products and two sizes of transportation units. Both models were re-run with updated information for 3 periods using an 16x Intel Xeon(R) CPU E5620 2400 GHz processor with 11,7 GB RAM. Both models were coded using Python and Pyomo (Hart et al., 2012) and the problem given in Table 3 is solved directly by Gurobi 5.6.3 (Gurobi, 2014).

A constrained non-linear problem was solved using the Ipopt non-linear solver (Wächter and Biegler, 2006) to generate each scenario tree for the stochastic model implementation. The demand forecasts for the deterministic model were received using the exponential smoothing method. During the analyzed time period the demand for the selected range of products was decreasing, which can be seen on the graph representing the value of the total demand (Figure 4).

The demand data for the first six months and the information about available inventory and placed orders by the end of the sixth month were utilized to find the order quantities for period 1 using the deterministic and the stochastic model. To find the order quantities for period 2 the demand information was updated and the information about the order placed in period 1 was utilized. The same procedure was used to find an order quantities in period 3. Then the demand information for the last four months was used to compute inventory, backorder and emergency costs. It means that both the stochastic model and the deterministic model were run three times to determine order quantities for periods 1, 2 and 3.

Figure 4: Value of the total demand during the analyzed time horizon

The multi-stage stochastic model outperformed the deterministic model with 51.4% lower total cost. Application of the deterministic model lead to significantly higher inventory costs (see Figure 5), whereas the stochastic model application lead to a slight increase in use of emergency supplies.

One run of the stochastic model took on average 39471.1 seconds of CPU time to be solved. Of that, 39279.9 seconds of that were used to generate a scenario tree and 153.2 seconds were used to solve an extensive form of the stochastic model. The deterministic model was solved in 0.56 seconds of CPU time on average. In the case company such a model is supposed to be solved once in a month, hence the time to solve a stochastic

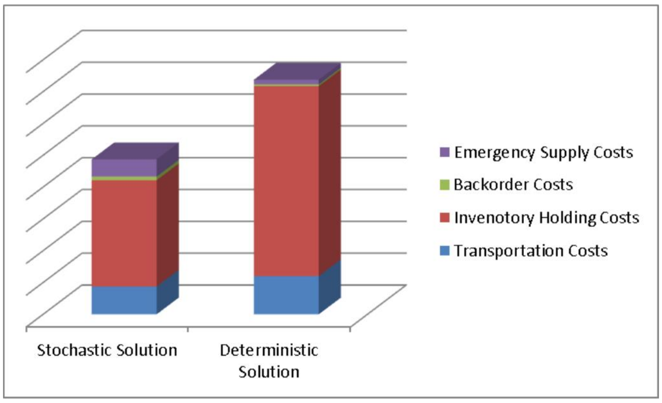

Figure 5: Distribution of the total cost

model is reasonable. Nevertheless, in other applications a faster method of scenario generation can be required.

6 Conclusions

In this paper we propose a model formulation and a solution approach for the coordinated capacitated lot-sizing problem (CCLSP) with stochastic demand and demonstrate its importance with real-world data. Whereas the deterministic version of the problem has been considered by a number of researches (see Robinson et al. (2009) for the most recent overview), the CCLSP with stochastic demand has not been addressed in the literature before.

Inclusion of joint setup costs makes the CCLSP different from the stochastic capacitated lot-sizing problem (SCLSP) (Tempelmeier, 2011; Helber et al., 2013) and, together with the aim to establish a concrete production plan rather than a policy, distinguishes it from the stochastic economic lot-scheduling problem (SELSP) (Winands et al., 2011).

The considered problem is characterized by stochastic, non-stationary in mean and correlated, general distribution product demands. Compared to the classical coordinated lot-sizing problem, the proposed models include a set of important changes reflecting the real world case: back-order costs are considered as part of the total cost function; emergency supplies are allowed and not limited; the setup costs are dependent on the number and size of used transportation units rather than on the inclusion of a particular item in an order.

A deterministic and a stochastic model formulation to deal with such a problem are presented and implemented on the data from the Norwegian company Stokke, designing and distributing products for children. The combined moment matching and scenario reduction method (Rubasheuski et al., 2014b) is used to create scenario trees to implement the stochastic model, and the exponential smoothing method was used to produce the demand forecasts for the deterministic model. The multi-stage stochastic model outperformed the deterministic model with 51.4% lower total cost when applied to a 3 period real data case from Stokke. 153.2 seconds of CPU time were used to solve an extensive form of the stochastic model with 731 scenarios and 0.56 seconds were used to solve the deterministic model. However, significant time required to generate a scenario tree could restrict the implementation of the stochastic model formulation in other applications.

We have demonstrated that a stochastic model can be used for this important class of lot sizing problems. Scenario generation takes significant, but tractable computational effort and the problem itself can be solved in reasonable time for a realistic sized instance. We have shown an example using real-world data where significant savings are possible using the stochastic model we have proposed.

References

Beraldi, P., Ghiani, G., Guerriero, E., and Grieco, A. (2006). Scenario-based planning for lot-sizing and scheduling with uncertain processing times. *International Journal of Production Economics*, 101(1):140–149.

Brandimarte, P. (2006). Multi-item capacitated lot-sizing with demand uncertainty. *International Journal of Production Research*, 44(15):2997–3022.

Delaert, N. and Melo, M. (1998). Make-to-order policies for a stochastic lot-sizing problem using overtime. *International Journal of Production Economics*, 56-57:79–97.

Gurobi (2014). Gurobi optimizer: Quick start guide.

Harris, F. (1913a). How many parts to make at once. *Factory: The Magazine of Management*, 10:135–136, 152.

Harris, F. (1913b). How much stock to keep on hand. *Factory: The Magazine of Management*, 10:240–241, 281–284.

Hart, W. E., Laird, C., Watson, J.-P., and Woodruff, D. (2012). *Pyomo - Optimization Modeling in Python*. Springer.

Helber, S., Sahling, F., and Schimmelpfeng, K. (2013). Dynamic capacitated lot sizing with random demand and dynamic safety stocks. *OR Spectrum*, 35:75–105.

Kaut, M. and Wallace, S. (2007). Evaluation of scenario-generation methods for stochastic programming. *Pacific Journal of Optimization*, 3(2):257–271.

Khouja, M. and Goyal, S. (2008). A review of the joint replenishment problem literature: 1989-2005. *European Journal of Operational Research*, 186:1–16.

King, A., Wallace, S., and Kaut, M. (2012). Scenario-tree generation. In *Modelling with Stochastic Programming*, pages 77–102. Springer.

Lee, L. and Chew, E. (2005). A dynamic joint replenishment policy with auto-correlated demand. *European Journal of Operational Research*, 165:729–747.

Levi, R., Pal, M., Roundy, R., and Shmoys, D. (2005). Approximation algorithms for stochastic inventory control models. *Manufacturing & Service Operations Management*, 7(1):82–85.

Levi, R. and Shi, C. (2013). Approximation algorithms for the stochastic lot-sizing problem with order lead times. *Operations Research*, 61(3):593–602.

Li, H. and Thorstenson, A. (2013). A multi-phase algorithm for a joint lot-sizing and pricing problem with stochastic demands. *International Journal of Production Research*, 52(8):2345–2362.

Robinson, P., Narayanan, A., and Sahin, F. (2009). Coordinated deterministic dynamic demand lot-sizing problem: A review of models and algorithms. *Omega*, 37:3–15.

Rubasheuski, U., Oppen, J., and Woodruff, D. (2014a). Minimization of transportation and inventory costs in a multi-product probabilistic demand environment. Technical report, Molde University College: Specialized University in Logistics.

Rubasheuski, U., Oppen, J., and Woodruff, D. (2014b). Multi-stage scenario generation by the combined moment matching and scenario reduction method. *Operations Research Letters*, 42:374–377.

Ruszczynski, A. and Shapiro, A. (2003). *Stochastic Programming*, volume 10 of *Handbooks in Operations Research and Management Science*. Elsevier.

Smith, J. and Tan, B. (2013). *Handbook of Stochastic Models and Analysis of Manufacturing System Operations*. International Series in Operations Research & Management Science. Springer.

Tempelmeier, H. (2011). A column generation heuristic for dynamic capacitated lot sizing with random demand under a fill rate constraint. *Omega*, 39:627–633.

Wächter, A. and Biegler, L. (2006). On the implementation of a primal-dual interior point filter line search algorithm for large-scale nonlinear programming. *Mathematical Programming*, 106:25–57.

Winands, E., Adan, I., and van Houtum, G. (2011). The stochastic economic lot scheduling problem: A survey. *European Journal of Operational Research*, 210(1):1–9.

Paper 4

Stochastic capacitated lot-sizing problems: a review of models and solution methods

Stochastic capacitated lot-sizing problems: a review of models and solution methods

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Abstract

The present literature review focuses on the stochastic capacitated lot-sizing problem (SCLSP). The SCLSP deals with make-to-stock production of multiple items sharing limited resources under stochastic demand over a finite time horizon divided into discrete time periods. A typical solution for such a problem would be a fixed production plan for the next period. We present an overview of all available models for the uncoordinated SCLSP and develop a new formulation for the coordinated SCLSP together with an indication of areas for future research.

Keywords: Inventory Theory; Stochastic Capacitated Lot-Sizing Problem;Coordinated Stochastic Capacitated Lot-Sizing Problem

1 Introduction

The problem of trading off inventory costs against ordering (transportation) costs has been in focus of researchers since F.Harris (1913) introduced the single item economic order quantity (EOQ) model. In this paper we present an overview of the literature on stochastic capacitated lot sizing problems for multiple items. These problems are characterized by a finite time horizon divided into discrete time periods, period-specific stochastic demands for multiple items, a target service level and limited shared resources.

In their survey Sox et al. (1999) considered the stochastic capacitated lot sizing problem as a version of a more general stochastic lot scheduling problem. The paper mainly considered the literature on the stochastic economic lot scheduling problem (SELSP), indicating only one paper dealing with the stochastic capacitated lot sizing problems (SCLSP) (Sox and Muckstadt, 1999). To the author's knowledge there are not other reviews of literature on stochastic capacitated lot sizing problems for multiple items.

Following and extending the taxonomy of deterministic dynamic demand lot-sizing problems presented by Robinson et al. (2009), we distinguish between coordinated (CSCLSP) and uncoordintated (SCLSP) versions of the problem. The CSCLSP is solved to minimize the total costs, which include joint setup costs induced whenever any of the items from the product group is produced, individual setup costs and inventory costs. In the SCLSP joint setups are not considered.

This review focuses on the development of formulations and solution methods for the stochastic capacitated lot sizing problems for multiple items over the last decade. We first give a short and limited overview of the literature on related problems, such as the stochastic joint replenishment problem (SJRP) and the stochastic economic lot scheduling problem (SELSP). In section 3 we discuss available formulations and solution methods of the uncoordinated and the coordinated stochastic capacitated lot sizing problems for multiple items. We also present a new formulation for the CSCLSP in section 3.2 and discuss the direction for the future research in section 4 of this paper.

2 Related Problems

Both the stochastic joint replenishment problem (SJRP) and the stochastic economic lot scheduling problem (SELSP) are extensions of the EOQ model, determining a replenishment policy for a set of products that minimizes total ordering and inventory costs.

The SELSP deals with make-to-stock production. The problem setting includes multiple items sharing common resources with limited capacity, under random demands, possibly random setup and production times. Solving the problem, one aims to establish a set of rules (policy) to manage a production process, rather than a concrete production plan. The policy describes for each possible state of the system, whether the production of the current item should be continued, whether the production of another item should begin, or whether the production should be terminated for a while. Winands et al. (2011) presented the most recent overview of the literature on the topic.

Another area of research, closely related to the stochastic capacitated lot sizing problem, is well studied within the frame of the stochastic joint replenishment problem (SJRP). In the classic SJRP one is interested to minimize total inventory holding and ordering costs, where the ordering costs include both individual and joint setup costs.

The two most common approaches for solving a stochastic JRP include implementation of a periodic review policy (Aksoy and Erenguk, 1988) or a can-order policy (Balintfy, 1964). Implementing the first policy one will find the time period between two reviews and the upper limit for inventory on hand. The order size in each particular order will stay unknown. Minimizing the total ordering and inventory costs under the can-order-policy, one will utilize a must-order level, a can order level and an up-to inventory level for each of the items. In this case neither the order size nor the time between two consecutive orders is known in advance. Khouja and Goyal (2008) presented the most recent review of the literature on the JRP, including a section on the SJRP.

3 Formulations of the Stochastic Capacitated Lot-Sizing Problem

3.1 The Uncoordinated Stochastic Capacitated Lot-Sizing Problem (SCLSP)

The stochastic uncoordinated capacitated lot-sizing problem (SCLSP) for multiple items was first analyzed by Sox and Muckstadt (1999). They developed a model formulation for the ”stochastic lot-scheduling problem as a finite-horizon, discrete-time, production and inventory problem with multiple products and random demand”. The authors have proposed a heuristic to solve the problem, capable to find a production plan, minimizing the total cost. Since the solution to the problem is not a policy but a fixed production plan, the common practice is to refer to the problem as to the SCLSP.

Table 1: Notation

Sets	
$\mathcal{N}$	set of products
$\mathcal{T}$	set of time periods to consider
$\mathcal{J}$	set of set of transport unit sizes
Parameters	
h_n	one period inventory holding cost for each unit of item $n \in \mathcal{N}$
s_n	setup cost for item $n \in \mathcal{N}$
S	joint setup cost
$d_{n,t}$	demand for item $n \in \mathcal{N}$ in period $t \in \mathcal{T}$
r_n	unit processing time for item $n \in \mathcal{N}$
r'_n	setup time for item $n \in \mathcal{N}$
R_t	available capacity in time period $t \in \mathcal{T}$
c	cost of production source
β^*_n	target fill rate per cycle for product $n \in \mathcal{N}$
$q_{n,t}$	lot size of item $n \in \mathcal{N}$ in period $t \in \mathcal{T}$
M	sufficiently large number
l	length of the lead time as number of time periods, integer
$o_{n,t}$	order for product n which was placed at time t, $n \in \mathcal{N}$, $t \in \{2\text{-}l, .., 0\}$
v_n	volume per unit of product n, $n \in \mathcal{N}$
u_j	capacity of the transport unit of size index j, $j \in \mathcal{J}$
w_j	transportation cost per transport unit of size index j, $j \in \mathcal{J}$
w^{em}	transportation cost per cubic meter from emergency supply source
f_n	one period backorder cost for product n, $n \in \mathcal{N}$
f_n^{max}	limit for backorder quantity of product n, $n \in \mathcal{N}$
$i_{n,0}$	initial inventory of product n, $n \in \mathcal{N}$

Variables	
$y_{n,t}$	lot size of product $n \in \mathcal{N}$ in period $t \in \mathcal{T}$

Table 1: (continued)

$y_{n,t,p}$	lot size of product $n \in \mathcal{N}$ in period $t \in \mathcal{T}$ to be consumed in period $p \in \mathcal{T} : p \geq t$
I_{nt}	net inventory for product $n \in \mathcal{N}$ at the end of period $t \in \mathcal{T}$
I_{nt}^{end}	backlog of product $n \in \mathcal{N}$ at the end of period $t \in \mathcal{T}$
I_{nt}^{prod}	backlog of product $n \in \mathcal{N}$ after production in period $t \in \mathcal{T}$, but before demand occurrence
F_{nt}	backorders of product $n \in \mathcal{N}$ in period $t \in \mathcal{T}$
l_{nt}	number of periods since the last setup (product $n \in \mathcal{N}$, period $t \in \mathcal{T} \cup \{0\}$)
$\gamma_{n,t} \in \{0, 1\}$	variable denoting if product $n \in \mathcal{N}$ was produced in period $t \in \mathcal{T}$ ($\gamma_{n,t} = 1$)
$\alpha_t \in \{0, 1\}$	variable denoting if any of the products was produced in period $t \in \mathcal{T}$ ($\alpha_t = 1$)
ω_{nt}	indicates if product $n \in \mathcal{N}$ was produced in period $t + 1, \ \forall t \in \mathcal{T}$ ($\omega_{nt} = 1$)
$Q_{j,t}$	number of transport units of size index j used for orders placed at the beginning of period t, $j \in \mathcal{J}, \ t \in \{1, .., T\}$
$A_{n,t}$	quantity of product n supplied during period t using the emergency source, $n \in \mathcal{N}, \ t \in \{L, .., T{+}L\}$

We present the model formulation, developed by Sox and Muckstadt (Table 2), with use of notation presented in Table 1, and where $E\left[g\left(x\right)\right]$ is expected value of function $g\left(x\right)$.

Table 2: Sox and Muckstadt model formulation for the SCLSP

$$min \sum_{n \in \mathcal{N}} \sum_{t \in \mathcal{T}} \left(s_n \gamma_{n,t} + h_n E \left[\sum_{tt=1}^{t} y_{n,tt} - \sum_{tt=1}^{t} d_{n,tt} \right]^+ + \right.$$

$$\left. + f_n E \left[\sum_{tt=1}^{t} d_{n,tt} - \sum_{tt=1}^{t} y_{n,tt} \right]^+ + c\left(r_n y_{n,t} + r'_n \gamma_{n,t} \right) \right) \quad (1)$$

Subject to:

$$\sum_{tt=1}^{t-1} y_{n,tt} \leq \sum_{tt=1}^{t} y_{n,tt}, \ \forall n \in \mathcal{N}, \ t \in \mathcal{T} \quad (2)$$

$$\sum_{tt=1}^{t} y_{n,tt} - \sum_{tt=1}^{t-1} y_{n,tt} \leq M \gamma_{n,t}, \ \forall n \in \mathcal{N}, \ t \in \mathcal{T} \quad (3)$$

Table 2: (continued)

$$\sum_{n \in \mathcal{N}} \left(r_n y_{n,t} + r'_n \gamma_{n,t} \right) \leq R, \ \forall\, n \in \mathcal{N} \tag{4}$$

$$y_{n,t} \geq 0, \ \forall\, n \in \mathcal{N}, \ t \in \mathcal{T} \tag{5}$$

$$\gamma_{n,t} \in \{0, 1\}, \ \forall\, n \in \mathcal{N}, \ t \in \mathcal{T} \tag{6}$$

Brandimarte (2006) proposed a plant-location model formulation for the SCLSP with multiple products. The problem setting included multiple items with stochastic demand, finite and discrete time horizon, item- dependent setup times, and lost sales in case of insufficient inventory.

For the sake of compactness we present the formulation for the problem assuming deterministic demand (Table 3).

Table 3: A plant-location model formulation for SCLSP

$$min \sum_{n \in \mathcal{N}} \sum_{t \in \mathcal{T}} \sum_{p \in \mathcal{T}: p \geq t} h_n \left(p - t \right) y_{ntp} + \sum_{n \in \mathcal{N}} \sum_{t \in \mathcal{T}} s_n \gamma_{nt} \tag{7}$$

Subject to:

$$\sum_{t \in \mathcal{T}: t \leq p} y_{ntp} \geq d_{np}, \ \forall\, n \in \mathcal{N}, \ p \in \mathcal{T} \tag{8}$$

$$y_{ntp} \leq d_{np} \gamma_{nt}, \ \forall\, n \in \mathcal{N}, \ t \in \mathcal{T}, \ p \in \mathcal{P} : p \geq t \tag{9}$$

$$\sum_{n \in \mathcal{N}} \sum_{p \in \mathcal{T}: p \geq t} r_n y_{ntp} + \sum_{n} r'_n s_{nt} \leq R_t, \ \forall t \in \mathcal{T} \tag{10}$$

$$y_{ntp} \geq 0, \ \forall\, n \in \mathcal{N}, \ t \in \mathcal{T}, \ p \in \mathcal{P} : p \geq t \tag{11}$$

$$\gamma_{nt} \in \{0, 1\}, \ \forall\, n \in \mathcal{N}, \ t \in \mathcal{T} \tag{12}$$

Brandimarte used a node-based stochastic programming approach to tackle the stochastic variant of the problem. The model formulation is then extended by inclusion of variables $x_n^{[g]}$, determining the lost sales of product $n \in \mathcal{N}$ at node g of the scenario tree. Constraint set 8 is replaced by a balance constraint for each product at each node. Big-M-type constraint 9 is also modified in order to reflect the stochastic nature of the problem. To solve the problem Brandimarte successfully applied a time-sweep-based heuristic solving relaxed MILP subproblems using CPLEX 8.0.

Another approach to tackle the SCLSP was introduced by Tempelmeier and Herpers (2010) and further developed by Tempelmeier (2011). They introduced a target service level β_n, limiting the expected number of backordered items in each production cycle. A model formulation for the $SCLSP_\beta$ is presented in Table 4

Table 4: A model formulation for the $SCLSP_\beta$

$$\textit{min} \sum_{n\in\mathcal{N}} \sum_{t\in\mathcal{T}} \left(s_n\gamma_{nt} + h_n E\left\{[I_{nt}]^+\right\}\right) \tag{13}$$

Subject to:

$$I_{n,t-1} + y_{nt} - d_{nt} = I_{nt},\ \forall\, n \in \mathcal{N},\ t \in \mathcal{T} \tag{14}$$

$$y_{nt} - M\gamma_{nt} \leq 0,\ \forall\, n \in \mathcal{N},\ t \in \mathcal{T} \tag{15}$$

$$\sum_{n\in\mathcal{N}} r_n y_{nt} \leq R_t,\ t \in \mathcal{T} \tag{16}$$

$$I_{nt}^{prod} = -\left[I_{n,t-1} + q_{nt}\right]^-,\ \forall\, n \in \mathcal{N},\ t \in \mathcal{T} \tag{17}$$

$$I_{nt}^{end} = -\left[I_{nt}\right]^-,\ \forall\, n \in \mathcal{N},\ t \in \mathcal{T} \tag{18}$$

$$F_{nt} = I_{nt}^{end} - I_{nt}^{prod},\ \forall\, n \in \mathcal{N},\ t \in \mathcal{T} \tag{19}$$

$$l_{nt} = \left(l_{n,t-1} + 1\right) \cdot \left(1 - \gamma_{n,t}\right),\ \forall\, n \in \mathcal{N},\ t \in \mathcal{T} \tag{20}$$

$$l_{n0} = -1,\ \forall\, n \in \mathcal{N} \tag{21}$$

Table 4: (continued)

$$\omega_{nt} = \gamma_{n,t+1}, \ \forall n \in \mathcal{N}, \ t \in \mathcal{T} \setminus \left\{ t_{|\mathcal{T}|} \right\} \tag{22}$$

$$\omega_{n,|\mathcal{T}|} = 1, \ \forall n \in \mathcal{N} \tag{23}$$

$$1 - \frac{E\left\{\sum_{j=t-l_{nt}}^{t} F_{nj}\right\}}{E\left\{\sum_{j=t-l_{nt}}^{t} d_{nj}\right\}} \geq \beta_n^*, \ \forall n \in \mathcal{N}, \ t \in \{\mathcal{T} \, | \omega_{nt} = 1\} \tag{24}$$

$$y_{nt} \geq 0, \ \forall n \in \mathcal{N}, \ t \in \mathcal{T} \tag{25}$$

$$\gamma_{nt} \in \{0, 1\}, \ \forall n \in \mathcal{N}, \ t \in \mathcal{T} \tag{26}$$

The model formulation is based on the assumption of a fixed production plan for the whole time horizon, and this production plan should be determined at the beginning of the planning horizon. Tempelmeier and Herpers (2010) implemented an ABC$_\beta$ heuristic to deal with the problem and Tempelmeier (2011) used a column generation heuristic. Comparison of numerical results for both heuristics (Tempelmeier, 2011) indicates that the column generation heuristic outperformed the ABC$_\beta$ heuristic in 97.95% of all solved problem instances, however it was not able to solve 148 of 2804 problem instances. Implementing the column generation Tempelmeier (2011) generated a number of production plans for each of the products, and then selected the best combination of them in the master problem.

Helber et al. (2013) changed the type of service level in the model formulation proposed by Tempelmeier and Herpers (2010). They used $\sigma -$ service level representing the expected percentage of the maximum possible demand-weighted waiting time that the customers of product n are protected against:

$$\sigma = 1 - \frac{\sum_{t \in \mathcal{T}} E\{F_{nt}\}}{\sum_{t \in \mathcal{T}} \left(t_{|\mathcal{T}|} - t + 1\right) E\{d_{nt}\}} \tag{27}$$

Helber et al. (2013) proposed to use a fix-and-optimize heuristic to solve the problem. They used a product-oriented decomposition and absence of improvement on last iteration as the termination criterion.

The same type of heuristic was used by Tempelmeier and Hilger (2015) to solve the linearized version of the problem. Non-linear functions of the expected inventory and backorders were replaced by piecewise linear functions. The model was also modified to account for different types of service level as well as for possible setup carry overs.

3.2 The Coordinated Stochastic Capacitated Lot-Sizing Problem (CSCLSP)

Solving a coordinated SCLSP under stochastic demand, one aims to determine a replenishment schedule (Robinson et al., 2009). This schedule contains information about order quantities and time intervals between two consequent orders. In addition to individual setup costs considered in the uncoordinated version of the problem, joint setup costs are included in the objective function.

Despite the fact that the coordinated lot-sizing problem with deterministic demand is broadly analyzed (see, e.g., Robinson et al. (2009)), literature on the stochastic version of the problem can hardly be found. A modified version of the CSCLSP was discussed by Rubasheuski et al. (2015).

The authors replaced joint setup costs and item specific setup costs with transportation costs which depend on the number of transport units of given capacities being ordered. Making use of parameters and variables shown in Table 1 one can express the transportation (setup) costs as follows:

$$Transportation\ costs = \sum_{t=1}^{T} \sum_{j \in \mathcal{J}} Q_{j,t} \cdot w_j \tag{28}$$

In addition the authors considered backorder costs and emergency supply costs as a part of the objective function. A multi-stage stochastic programming model optimizing the objective function over $\mathcal{Z}$ possible scenarios was used to solve the problem (see Table 5).

Table 5: Multi-Stage Stochastic Programming model for the CSCLSP

$$\begin{aligned} min\, C = \sum_{z \in \mathcal{Z}} p_z \cdot \Bigg(& \sum_{t=1}^{T} \sum_{j \in \mathcal{J}} Q_{j,t,z} \cdot w_j \\ & + \sum_{t=l+1}^{T+l} \sum_{n \in \mathcal{N}} \frac{I_{n,t\text{-}1,z} - F_{n,t\text{-}1,z} + y_{n,t\text{-}l,z} + I_{n,t,z}}{2} \cdot h_n \\ & + \sum_{t=l+1}^{T+l} \sum_{n \in \mathcal{N}} F_{n,t,z} \cdot f_n + \sum_{t=l}^{T+l} \sum_{n \in \mathcal{N}} A_{n,t,z} \cdot v_n \cdot w^{em} \Bigg) \end{aligned} \tag{29}$$

Subject to:

$$I_{n,l,z} - F_{n,l,z} - A_{n,l,z} = \sum_{t=1\text{-}l}^{0} o_{n,t} + \sum_{t=1}^{l} d_{n,t,z} + i_{n,0}\ , \ \forall n \in \mathcal{N}, z \in \mathcal{Z} \tag{30}$$

Table 5: (continued)

$$y_{n,t,z} + I_{n,t+l-1,z} + F_{n,t+l,z} + A_{n,t+l,z} = d_{n,t+l,z} + I_{n,t+l,z} + F_{n,t+l-1,z}, \quad \forall n \in \mathcal{N},\ t \in \{1,..,T\},\ z \in \mathcal{Z} \tag{31}$$

$$\sum_{n \in \mathcal{N}} y_{n,t,z} \cdot v_n \leq \sum_{j \in \mathcal{J}} Q_{j,t,z} \cdot U_j,\ \forall z \in \mathcal{Z},\ t \in \{1,..,T\} \tag{32}$$

$$X \in \mathcal{D}_{\mathcal{Z}} \tag{33}$$

$$y_{n,t,z} \geq 0, \forall n \in \mathcal{N},\ t \in \{1,..,T\},\ z \in \mathcal{Z} \tag{34}$$

$$F_{n,t,z} \leq f_n^{max}, \forall n \in \mathcal{N},\ t \in \{l,..,T+l\},\ z \in \mathcal{Z} \tag{35}$$

$$I_{n,t,z} \geq 0, F_{n,t,z} \geq 0,\ \forall n \in \mathcal{N},\ t \in \{l,..,T+l\},\ z \in \mathcal{Z} \tag{36}$$

$$Q_{j,t,z} \in \mathbf{N},\ \forall j \in \mathcal{J},\ t \in \{1,..,T\},\ z \in \mathcal{Z} \tag{37}$$

Constraint set (33) requires that the system of solution vectors X is *implementable*. This means that the first stage variables must have the same values for all scenarios. The second stage variables will be the same for all branches of the scenario tree starting from the same second-stage node, etc. Assume that there can be six different outcomes of the stochastic process on the second stage, i.e., each of the scenarios goes through one of the six nodes on the second stage. Hence, all scenarios going through the same node, should have the same values of the decision variable on the second stage. Such logic applies for all the stages and scenarios.

Table 6: An overview of models for the SCLSP

	Representation of a stochastic process	
Model type	Scenario Tree	Functions of expected values
Uncoordinated	Brandimarte (2006)	Sox and Muckstadt (1999) Tempelmeier and Herpers (2010) Tempelmeier (2011) Helber et al. (2013)

Table 6: (continued)

		Tempelmeier and Hilger (2015)
Coordinated	Rubasheuski et al. (2015)	

The presented review of available models for the SCLSP (Table 6 indicates that authors so far mostly concentrated on the uncoordinated version of the problem. Developing the model of Rubasheuski et al. (2015), we propose a scenario-based formulation for a classic version of the coordinated SCLSP under stochastic demand.

Table 7: Multi-Stage Stochastic Programming model for the classic CSCLSP

$$min\, C = \sum_{z\in\mathcal{Z}} p_z \cdot \left(\sum_{t\in\mathcal{T}} S\alpha_{t,z} + \sum_{n\in\mathcal{N}} \sum_{t\in\mathcal{T}} s_{n,z}\gamma_{n,t,z} + \right.$$
$$\left. + \sum_{n\in\mathcal{N}} \sum_{t\in\mathcal{T}} I_{n,t}h_n + \sum_{t\in\mathcal{T}} \sum_{n\in\mathcal{N}} F_{n,t,z} \cdot f_n \right) \quad (38)$$

Subject to:

$$I_{n,1,z} - F_{n,1,z} = y_{n,1,z} + d_{n,1,z} + i_{n,0}, \ \forall n \in \mathcal{N}, z \in \mathcal{Z} \quad (39)$$

$$y_{n,t,z} + I_{n,t\text{-}1,z} + F_{n,t,z} = d_{n,t,z} + I_{n,t,z} + F_{n,t\text{-}1,z},$$
$$\forall n \in \mathcal{N},\ t \in \{2,..,T\},\ z \in \mathcal{Z} \quad (40)$$

$$\sum_{n\in\mathcal{N}} y_{n,t,z} r_n \leq R_t \alpha_{t,z},\ \forall z \in \mathcal{Z},\ t \in \{1,..,T\} \quad (41)$$

$$y_{n,t,z} r_n \leq R_t \gamma_{n,t,z},\ \forall n \in \mathcal{N},\ z \in \mathcal{Z},\ t \in \{1,..,T\} \quad (42)$$

$$\gamma_{n,t,z} \leq \alpha_{t,z},\ \forall n \in \mathcal{N},\ z \in \mathcal{Z},\ t \in \{1,..,T\} \quad (43)$$

$$X \in \mathcal{D}_{\mathcal{Z}} \quad (44)$$

$$y_{n,t,z} \geq 0, \forall\, n \in \mathcal{N},\ t \in \{1,..,T\},\ z \in \mathcal{Z} \quad (45)$$

Table 7: (continued)

$$I_{n,t,z} \geq 0, F_{n,t,z} \geq 0, \forall n \in \mathcal{N},\ t \in \{I, .., T\},\ z \in \mathcal{Z} \quad (46)$$

$$\gamma_{n,t,z} \in \{0,\ 1\}, \forall n \in \mathcal{N},\ t \in \{I, .., T\},\ z \in \mathcal{Z} \quad (47)$$

$$\alpha_{t,z} \in \{0,\ 1\}, \forall t \in \{I, .., T\},\ z \in \mathcal{Z} \quad (48)$$

This formulation is an extension of the formulation presented by Federgruen et al. (2007) for the deterministic coordinated capacitated lot-sizing problem. It takes into account the possibility of backorder occurrence in stochastic demand settings, and tackles the stochastic nature of demand via scenario representation. This formulation will be suitable for scenarios with stationary and non-stationary, correlated and uncorrelated demands, and different values of other parameters.

4 Conclusion

The present paper has focused on models and solution methods for the stochastic capacitated lot-sizing problem. Following the taxonomy presented by Robinson et al. (2009), the literature on the coordinated and uncoordinated versions of the SCLSP was analyzed. Related problems have been discussed, including the most recent literature reviews, in section 2.

The current survey indicates that the majority of researchers have concentrated on development of model formulations and solution methods for the uncoordinated version of SCLSP. Authors have used two common methods to tackle the stochasticity of demand, either by solving a problem for a number of scenarios or by functions of expected inventory and backorder volumes. The only paper dealing with the coordinated SCLSP is presented by Rubasheuski et al. (2015) and deals with a special case of the problem.

As a contribution to the literature on the topic, a scenario based formulation for the classic version of CSCLSP is presented, based on the formulation for the deterministic version of the problem by Federgruen et al. (2007). Development of more sophisticated formulations and solution methods for the classic CSCLSP can be a promising direction for future research on the topic.

References

Aksoy, Y. and Erenguk, S. S. (1988). Multi-item inventory models with co-ordinated replenishments: A survey. *International Journal of Operations and Production Management*, 8 (1):63–73.

Balintfy, J. L. (1964). On a basic class of multi-item inventory problems. *Management Science*, 10 (2):287–297.

Brandimarte, P. (2006). Multi-item capacitated lot-sizing with demand uncertainty. *Internarional Journal of Production Research*, 44(15):2997–3022.

Federgruen, A., Meissner, J., and Tzur, M. (2007). Progressive interval heuristic for multi-item capacitated lot-sizing problems. *Operations Research*, 55(3):490–502.

Harris, F. (1913). How much stock to keep on hand. *Factory: The Magazine of Management*, 10:240–241, 281–284.

Helber, S., Sahling, F., and Schimmelpfeng, K. (2013). Dynamic capacitated lot sizing with random demand and dynamic safety stocks. *OR Spectrum*, 35:75–105.

Khouja, M. and Goyal, S. (2008). A review of the joint replenishment problem literature: 1989-2005. *European Journal of Operational Research*, 186:1–16.

Robinson, P., Narayanan, A., and Sahin, F. (2009). Coordinated deterministic dynamic demand lot-sizing problem: A review of models and algorithms. *Omega*, 37:3–15.

Rubasheuski, U., Oppen, J., and Woodruff, D. (2015). A stochastic programming approach to solve a coordinated capacitated stochastic dynamic demand lot-sizing problem with emergency supplies. Technical report, Molde University College: Specialized University in Logistics.

Sox, C. R., Jackson, P. L., Bowman, A., and Muckstadt, J. A. (1999). A review of the stochastic lot scheduling problem. *International Journal of Production Economics*, 62(3):181–200.

Sox, C. R. and Muckstadt, J. A. (1999). Optimization-based planning for the stochastic lot-scheduling problem. *IIE Transactions*, 29(5):349–357.

Tempelmeier, H. (2011). A column generation heuristic for dynamic capacitated lot sizing with random demand under a fill rate constraint. *Omega*, 39:627–633.

Tempelmeier, H. and Herpers, S. (2010). ABC_β - a heuristic for dynamic capacitated lot sizing with random demand under a fill rate constraint. *International Journal of Production Research*, 48:5181–5193.

Tempelmeier, H. and Hilger, T. (2015). Linear programming models for a stochastic dynamic capacitated lot sizing problem. *Computers & Operations Research*, 59:119–125.

Winands, E., Adan, I., and van Houtum, G. (2011). The stochastic economic lot scheduling problem: A survey. *European Journal of Operational Research*, 210(1):1–9.

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